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and

$$\int_0^1 (x^p + x^{-p}) \log(1-x) \frac{dx}{x} = \frac{\pi}{p} \cot p\pi - \frac{1}{p^2} \ldots\ldots(2),$$

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5624. (Prof. Sylvester, F.R.S.)—If X_j is to be a rational homogeneous function of the j^{th} order of i independent quantities $a, b, c, \ldots k, l$, and is to satisfy the partial differential equation

$$(A\delta_a + B\delta_b + \ldots + L\delta_l)\, X_j = 0,$$

where A, B, ... L are homogeneous linear functions of $a, b, \ldots l$, not absolutely independent, but connected by a single homogeneous linear equation with one another; prove that the number of arbitrary parameters in X_j will in general be the number of solutions in positive integers (zero counted in as positive) of the simultaneous equations

$$u_0 + u_1 + u_2 + \ldots + u_{i-1} = j;\quad \tfrac{1}{2}jr_0 + u_1 + 2u_2 + \ldots (i-1)\, u_{i-1} = (i-1)\,\tfrac{1}{2}jr_0 + v_1 = 1.$$

Or, again, suppose $[(a-b)\,\delta_c + (b-c)\,\delta_a + (c-a)\,\delta_b]\, X_2 = 0$, the complete value of X_2, viz., $k(a^2+b^2+c^2) + k'(ab+ac+bc)$, contains 2 parameters k, k', which will be found to agree with the rule; and so, if in the above we write X_{2i} or X_{2i-1} in place of X_2, it will be found (still in accordance with the rule) that there will be $i+1$ arbitrary parameters in each of these forms. 55

5708. (Prof. Seitz, M.A.)—Within a given circle a line is drawn at random in position and length; show that (1) the chance that the line intersects a given diameter is π^{-1}; and (2) the average length of the line is $\frac{3}{16}\pi r$. 107

5791. (W. H. H. Hudson, M.A.)—Let AB, BC, CD be equal arcs of a circle, TA, TD be the tangents at A, D; AB, DC meet in Y; XB, XC, the tangents at B, C, meet AD, produced in EF; show that, when B, C, D move up to A, the ultimate ratio of the triangles TAD, XEF, YAD, are as 27 : 25 : 18. 30

5889. (J. J. Walker, M.A.) — Show that $l\alpha + m\beta + n\gamma$ will be a side of the cone having origin as vertex, and the circle through the terms of α, β, γ as base, if

$$l^{-1}T^2(\beta-\gamma) + m^{-1}T^2(\gamma-\alpha) + n^{-1}T^2(\alpha-\beta) = 0;$$

and hence deduce immediately the existence of the second system of cirular sections parallel to the plane through the terms of $\alpha^{-1}, \beta^{-1}, \gamma^{-1}$. 61

5901. (Prof. Sylvester, F.R.S.)—Let μ points be given on a cubic curve. Through them draw any curve (simple or compound) of degree ν; the remaining $3\nu - \mu$ (say μ') points may be termed a first residuum to the given ones. Through these μ' points draw any curve of degree ν'; the remaining $3\nu' - \mu'$ points may be termed a residuum of the second order to the given ones; and in this way we may form at pleasure a series of residua of the third, fourth, and of any higher order. If μ is of the form $3i-1$, a residuum of the first or any odd order; and if μ is of the form $3i+1$, a residuum of the second or any even order in such series, may be made to consist of a single point, which I call *residual* of the original μ points. Prove that any such residual is dependent wholly and solely on the original μ points, being independent of the number, degrees, and forms of the successive auxiliary curves employed to arrive at it..... 34

5926. (Prof. Sylvester, F.R.S.) — By the potential work (in respect to an arbitrary centre) of a body describing any orbit, plane or twisted, understand what the work would be were it moving in a circle about such centre under the influence of the

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forces actually soliciting it; and by the residual work understand the excess of the actual stored-up work over such potential work. Also in general let an acceleration in any quantity (q) signify its second time-derivative $\left(\frac{d^2q}{dt^2}\right)$. Prove that the residual work of a particle of unit mass at each moment of time, in respect to any centre, is equal to the acceleration in the square of the radius vector drawn to such centre..... 65

5947. (Prof. Seitz, M.A.)—Two points are taken at random within a circle, one on each side of a chord which divides the circle into two segments whose areas are u_1, u_2; show that the chance that the chord drawn through the points is less than a chord which cuts an area u from the circle, is

$$\frac{u^2}{u_1 u_2}, \text{ when } u < u_1 < u_2, \text{ and } \frac{2u-u_1}{u_2}, \text{ when } u > u_1. \text{ .....} \quad 59$$

5962. (Prof. Cochez.) — Trouver la courbe dont le rapport du rayon de courbure à la normale est constant et égal à m. 72

5968. (The late Prof. Benjamin Peirce, F.R S.) — If two bodies revolve about a centre, acted upon by a force proportional to the distance from the centre, and independent of the mass of the attracted body, prove that each will appear to the other to move in a plane, whatever be the mutual attraction..... 111

5973 & 6002. (J. L. Mackenzie, B.A.)—Solve the functional equation $\phi(ax+\beta) = \gamma\phi(x) + u$, where a, β, γ are constants and u is a known function of x; and, as particular cases, solve the equations

$$\phi(a-x)\phi(x) + b[\phi(a-x) + \phi(x)] = c^2, \ \phi(mx) = \phi(x) + ax\log x. \quad 28$$

5976. (R. Tucker, M.A.) — P_1, P_2, P_3; (Q_1, Q'_1), (Q_2, Q'_2), (Q_3, Q'_3) are the respective denominators and numerators of the ultimate (proper fraction) convergents, which are obtained from making the roots of the equation $x^3 + qx + r = 0$ the quotients of a series of continued fractions; shew that

$$\frac{Q_1Q'_1}{P_1} + \frac{Q_2Q'_2}{P_2} + \frac{Q_3Q'_3}{P_3} = \frac{1}{P_1} + \frac{1}{P_2} + \frac{1}{P_3}. \text{ .....} \quad 69$$

5983. (Prof. Sylvester, F.R S.)—The i^{th} involute to a circle being defined as a cyclode of the i^{th} order, and a symmetrical cyclode as one which can be divided into two precisely equal and similar parts; prove that there are as many distinct species of symmetrical cyclodes of the order $2n$ possessing the property that the length of the arc of any one of them is connected by an algebraical equation with the length of the corresponding arc of its pedal, as there are ways of breaking up the number n into parts all unequal to one another..... 79

6000. (W. S. B. Woolhouse, F.R.A.S.)—The separate probabilities of three partially dependent events are given as well as the probabilities of the joint occurrence of every two events and also the joint occurrence of all three. In a proposed case, however, it is known that two of the events have failed, but it is unknown which; determine the probability of the failure also of the remaining event..... 26

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6005. (The Editor.)—If three lengths are taken at random, and the limits be the same (and any magnitude) for all three, show that the respective probabilities of being able to form with them (1) a triangle of any kind, (2) an obtuse-angled triangle, are

$$p_1 = \tfrac{1}{4},\ p_2 = \tfrac{1}{4}\pi \ldots\ldots 22$$

6008. (Prof. Sylvester, F.R.S.)—Prove that the equation

$$x^2y + x^2z + y^2x + y^2z + z^2x + z^2y = 0$$

is insoluble by positive or negative integer values of x, y, z..... 99

6019 & 6088. (T. R. Terry, M.A.)—If $a < 1$ and m and n are zero or any positive integers, and

$$f(n,m) = \frac{(2n)!}{2^{2n}n!\,n!}\left\{1 - \frac{n-m}{n+1}na^2 + \frac{(n-m)(n-m+1)}{n+1\,.\,n+2}\cdot\frac{m(m-1)}{1\,.\,2}a^4 - \ldots\right\},$$

prove that $4a^2f(n,m) = -f(n-1,m+1) + 2(1+a^2)f(n-1,m) - (1-a^2)^2 f(n-1,m-1)$.....(1),

$$\frac{df(n,m)}{da} = 4a\frac{(n-m)(n-m+1)}{2n+1}f(n+1,m-1) - 2a(n-m)f(n,m-1)\ldots(2),$$

$$f(n,m+1) = (1+a^2)f(n,m) - \frac{4a(n-m)}{2n+1}f(n+1,m)\ldots\ldots(3).\ 27$$

6034. (Prof. Sylvester, F.R.S.)—

$\left.\begin{array}{llll} U_1, & U_2, & \ldots & U_i \\ V_1, & V_2, & \ldots & V_j \\ W_1, & W_2, & \ldots & W_k \\ \ldots & \ldots & \ldots & \ldots \\ \ldots & \ldots & \ldots & \ldots \end{array}\right\}$ are homogeneous numerical linear functions of n variables; and $i+j+k+\ldots = n$. Any i homogeneous linear functions, in which the coefficients are all integer, are taken of the U's, j of the V's, k of the W's, and so on. Let R be the resultant of these n new linear functions of the n variables. Investigate a rule for determining the greatest common measure of the infinite number of values of R that can be thus obtained 71

6037. (Prof. Crofton, F.R.S.)—Prove that—

$e^{x+D}\,F(x) = e^i e^x\,F(x+1)$, and $e^{a(x+D)}\,Fx = e^{ia}e^{ax}\,F(x+a)$.. 41

6066. (Prof. Wolstenholme, M.A.) — A heavy particle moves on the interior of a smooth paraboloid whose axis is vertical, and z_1, z_2 are its least and greatest vertical heights above the vertex, ρ the radius of absolute curvature of its path when at a vertical height z: prove that

$$\frac{4a(a+z)^3}{z}\cdot\frac{(z_1+z_2-z)^2}{\rho^2} = \frac{\{(z_1+z_2-z)[a(z_1+z_2-z)+z_1z_2]^2 + az_1z_2(a+z)^2\}^3}{z(z_1+z_2-z)^2[a(z_1+z_2-z)+z_1z_2]^2 + az_1^2z_2^2(a+z)^3}.$$

..... 88

6072. (Prof. Cochez.)—Sur les côtés d'un triangle et dans le même sens on prend à partir des sommets de longueurs AD, BE, CF égales respectivement à $\frac{m\text{AB}}{n}$, $\frac{m\text{BC}}{n}$, $\frac{m\text{AC}}{n}$, et on joint les points D, E, F. Déterminer (1) le rapport $\frac{\text{DEF}}{\text{ABC}}$; (2) le rapport $\frac{a^2+b^2+c^2}{a'^2+b'^2+c'^2}$, a, b, c étant les côtés de ABC, et a', b', c' les côtés de DEF; (3) comment faut-il déterminer le rapport $\frac{m}{n}$ pour que le triangle DEF ait une surface minima ? 93

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6075. (Donald McAlister, D.Sc.)—A sphere full of water, is hung up by a string. Suddenly a slender uniform crack extends from one end of the vertical diameter to the other. Prove that the time of emptying (on the hypothesis of "parallel sections") is to the time of free fall from rest down the diameter, as the area of the sphere is to that of the lune. 36

6089. (D. Edwardes.)—If A, B, C are the angles of a triangle, prove that

$$2\{\sin^2(A-B)+\sin^2(B-C)+\sin^2(C-A)+\sin(A-B)\sin(C-A)\cos A + \sin(B-C)\sin(A-B)\cos B+\sin(C-A)\sin(B-C)\cos C\} = (\sin^2 A+\sin^2 B+\sin^2 C)(1-8\cos A\cos B\cos C)$$ 93

6091. (R. Tucker, M.A.) — If a point be taken on a certain line and normals be drawn from it to an ellipse, a circle can be drawn through the intersections of the normals with the ellipse. Find the envelop of this circle..... 85

6094. (Prof. Sylvester, F.R.S.) — 1. If the process of residuation, described in Question 2391, be applied to a quartic instead of to a cubic curve, prove that the minimum number of points in any *residuum* is 3, that being the number applicable to the case of an odd number of initial points. If the initial number is of the form $4i$, show that the minimum number in a residuum is 4, and when the initial number is of the form $4i+2$, 6, except for the case when $i=0$, when it is obviously only 2.

2. Prove that, if the number of initial points is odd, the 3 *residuals* are functions only of those points.

3. In general, if p be prime to μ, and p initial points be taken in a curve of the μ^{th} order, prove that the residuum may be reduced to consist of $\frac{1}{2}(\mu-1)(\mu-2)$ residual points, which residuals are all functions of the p given points only, being independent of the particular process of residuation applied to arrive at them..... 34

6101. (J. J. Walker, M.A.)—If one side of a thin bi-convex lens is turned towards the sun, a well-defined image may be formed on a small screen facing that side, say at a distance of α^{-1}. The same being done with the other side, and the distance being now α'^{-1}, show that, if μ is the refractive index, the power of the lens is given by

$$2(2\mu-1)\phi = -(\mu-1)(\alpha+\alpha')$$ 88

6106. (A. Martin, M.A.)—Show that the mean value of all the radius-vectors that can be drawn from one end of the major axis of a given ellipse to its circumference is $\frac{4b}{\pi e}\sin^{-1}e$..... 70

6128. (Prof. Tanner, M.A.)—The continued fraction

$$\frac{a}{b-}\,\frac{a}{b-}\cdots\frac{a}{b-}\,\frac{a}{b+x}$$ (r terms) being represented by u_r, find a, b so that, for all values of r, $u_r = u_{r+n}$, and verify the result by showing the form of u_r..... 65

6145. (C. Leudesdorf, M.A.)—Find three square numbers such that, if each be subtracted from the sum of the other two, and the remainders doubled, the resulting numbers shall be perfect squares..... 96

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6148. (D. Edwardes.) — A ring of weight W is just capable of sliding upon a rough semi-elliptic wire, and has attached to it two strings, which, passing through smooth small rings on the foci, support equal weights (P). If 2θ be the angle between the strings in a limiting position of equilibrium, prove that $W^2 \sin^2\theta + \mu^2 (W \sin\theta + 2Pe\cos\theta)^2 = e^2W^2$, where μ is the coefficient of friction, and e eccentricity of ellipse. 64

6151. (G. Turriff, M.A.)—If P, Q be two points on an ellipse such that the difference (α) of their excentric angles is constant, and if the ordinates at P and Q be produced to meet the auxiliary circle in P′ and Q′; show (1) that area of $\Delta P'CQ'$: area of $\Delta PCQ = a : b$, C being the centre; (2) that P′Q touches at its middle point the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = \cos^2 \frac{1}{2}\alpha$. 58

6154. (Prof. Sylvester, F.R.S.)—If the roots of a Quantic are all real, prove that the roots of its Hessian are all imaginary 108

6158. (Prof. Crofton, F.R.S.) — Find the mean distance of two points in a rectangle 101

6169. (Elizabeth Blackwood.)—If P, Q, R are random points within a sphere of which O is the centre; find the average volume of the tetrahedron OPQR. 33

6173. (Hugh McColl, B.A.)—If the statements a, b, f are true, then the statements c and x are true, or else the statements d, e, y are true. If the statement a is true, f false, and y true, then c is true and x false, or else d is false and e true. What inference may be drawn from these premises with respect to the truth or falsehood of the statements a, b, c, d, e, f, eliminating the statements x and y? 40

6186. (T. R. Terry, M.A.)—If $a < 1$, and m and n are any positive integers, prove that

$$1 + \frac{(m-1)\,m}{1\,.\,n+1}\frac{a^2}{1-a^2} + \frac{(m-2)\,(m-1)\,m\,(m+1)}{1\,.\,2\,.\,(n+1)\,(n+2)}\frac{a^4}{(1-a^2)^2} + \ldots = \frac{1}{(1-a^2)^{m-1}}$$
$$\times \left\{1 - \frac{n-m+1}{n+1}(m-1)\,a^2 + \frac{(n-m+1)\,(n-m+2)}{(n+1)\,(n+2)} \cdot \frac{(m-1)\,(m-2)}{1\,.\,2}\,a^4 - \ldots\right\}.$$

..... 104

6188. (Prof. Sylvester, F.R.S.) — If in any persymmetrical matrix of the n^{th} order, each column in succession calling its elements $c_1, c_2 \ldots c_n$, be replaced by $0, c_1, 2c_2 \ldots (n-1)\,c_n$, show that the sum of the n determinants thus formed is zero. For example, show that

$$\begin{vmatrix} 0 & b & c & d \\ a & c & d & e \\ 2b & d & e & f \\ 3c & e & f & g \end{vmatrix} + \begin{vmatrix} a & 0 & c & d \\ b & b & d & e \\ c & 2c & e & f \\ d & 3d & f & g \end{vmatrix} + \begin{vmatrix} a & b & 0 & d \\ b & c & c & e \\ c & d & 2d & f \\ d & e & 3e & g \end{vmatrix} + \begin{vmatrix} a & b & c & 0 \\ b & c & d & d \\ c & d & e & 2e \\ d & e & f & 3f \end{vmatrix} = 0.$$

..... 55

6196. (The Editor.)—If ABC, A′B′C′ be two triangles with their corresponding sides equal and parallel; prove that (1) of the parallelograms formed by joining corresponding vertices one is equal to the other two; and hence (2) prove Euc. I. 47..... 37

6203. (W. H. H. Hudson, M.A.)—If P be the population and W the wealth of an industrial community at a time t, (1) interpret $\frac{d}{dt}\left(\frac{W}{P}\right)$; and (2) show that, in such a community, if each

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person saves one-nth of his income, the average value of a person's work per unit of time is

$$n\frac{d}{dt}\left(\frac{W}{P}\right)+\frac{W}{P^2}\frac{dP}{dt} \quad \ldots\ldots 92$$

6204. (J. J. Walker, M.A.)—Referring to Quest. 5680, show that the conditions for the double contact of two conics — not expressible in terms of the invariants of the system—may be obtained in terms of coefficients of the three contravariant conics of the system. 74

6205. (W. J. C. Sharp, M.A.)—Prove that, according as the polar conic of a point on a cubic is an hyperbola, a parabola, or an ellipse, two lines, one line, or no line, can be drawn through the point, so as to be terminated by the curve and bisected at the point. 29

6207. (E. Anthony, M.A.)—In a plane triangle ABC, if $\frac{1}{2}\Sigma \sin^2 A \equiv \sigma$, and $\sigma - \sin^2 A = \sigma_1$, &c., prove that

$$\sigma_2\sigma_3+\sigma_3\sigma_1+\sigma_1\sigma_2 = \sin^2 A \sin^2 B \sin^2 C. \quad \ldots\ldots 56$$

6211. (R. E. Riley, B.A.)—Two points are taken on the axis of a parabola at equal distances on opposite sides of the focus, and from these points perpendiculars are drawn to a tangent. If SY be the perpendicular from the focus, show that the difference of the other two perpendiculars varies inversely as SY. 60

6212. (D. Edwardes.)—If ρ, R be the radii of absolute and spherical curvature at an ordinary point on a curve in space, and if ds, ds' be the elements of arcs at corresponding points on the locus of the centres of absolute and spherical curvature respectively, prove that $ds : ds' = d\rho : dR$. 62

6213. (A. Martin, M.A.) — A cylinder, of radius r, *rolls* down the surface of another one, radius R, resting on a horizontal plane, the surface of both cylinders and plane being rough enough to secure perfect rolling. Determine the circumstances of the motion, the point of separation, and the path of the axis of the upper cylinder. 44

6214. (G. Heppel, M.A.) — If the sides of a plane triangle be circular arcs of equal radius, find a simple necessary relation between the sides and angles of the triangles. 58

6216. (R. Tucker, M.A.) — O is the point of projection of two equi-horizontal-range particles, and (OP, OP'), (OQ, OQ') the normal chords and diameters of curvature at O respectively; prove that, if the initial velocity be the same for both, PQ' is parallel to QP'. 56

6218. (Prof. Sylvester, F.R.S.)—If

$$E = a\delta_b + 2b\delta_c + 3c\delta_d + 4d\delta_e \ldots, \qquad F = a\delta_c + 3b\delta_d + 6c\delta_e + 10d\delta_f \ldots,$$

$$G = a\delta_d + 4b\delta_e + 10c\delta_f + 20d\delta_g \ldots, \qquad H = a\delta_e + 5b\delta_f + 15c\delta_g + 35e\delta_h \ldots,$$

express $(E)^n$ as an algebraical function of E, F, G, H, &c. 99

6219. (Prof. Townsend, F.R.S.)—A thin uniform spherical shell being supposed to hold in free equilibrium a material particle, situated at its centre of form, by the attraction of its mass for a single power, direct or inverse, of the distance; determine the groups of powers for which the equilibrium of the particle is respectively stable and unstable. 114

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6226. (Prof. Matz, M.A.)—A post is a feet from the north wall, and b feet from the west wall of a room whose sides meet each other at right angles; find the sides of the largest rectangular table that can be passed so as to touch at the same time the walls and the post, the surface of the table to remain horizontal all the time 39

6232. (J. J. Walker, M.A.)—A system of forces being reduced to two, one of which passes through a fixed point and makes a given angle with the resultant of translation; show that the other has for its envelop an ellipse in the plane through the fixed point perpendicular to the axis of principal moment for that point; and that the ellipse becomes a circle when the fixed point is Poinsot's centre. 25

6235. (H. McColl, B.A.) — When, from the three implications, (1) $a'b + ab' : dx$, (2) $ax + by : c$, (3) $cd : y$, may we conclude that either x or y is true, but not both? 69

6238. (R. Knowles B.A., L.C.P.)—A paraboloid, of parameter m, when prevented from sliding down a plane whose inclination is α, just stands on its base without falling over; prove that the length of its axis is $9m \cot^2 \alpha$. 63

6241. (E. W. Symons, B.A.)—The roots of the equation

$$x^3 + \lambda x = (x^2 + 1)(2\lambda + \mu)^{\frac{1}{2}}$$

are all real and positive; prove that neither λ nor μ can be less than 9. 24

6243. (Prof. Sylvester, F.R.S.)—If $2f^2 + g^2$ is a prime and f an odd number, prove that $fy^2 + 2gxy - 2fx^2 = \pm 1$ is soluble in integers. 21

6245. (Prof. Cochez.)—Démontrer que l'enveloppe des axes des coniques tangentes à deux droites en deux points donnés est une parabole 49

6246. (Prof. Townsend, F.R.S.)—In the rotation, round its axis of figure, of a liquid spheroid of uniform density, held in permanent form by its own attraction for the ordinary law of the inverse square of the distance; show, from the equation of permanence, that

$$\frac{\omega^2}{4\pi\rho} = \frac{2}{3 \times 5}\lambda^2 - \frac{4}{5 \times 7}\lambda^4 + \frac{6}{7 \times 9}\lambda^6 - \frac{8}{9 \times 11}\lambda^8 + \&c.,$$

the several quantities involved having their usual significations. 30

6248. (The Editor.)—If two points P, Q be taken at random on the area of a vertical circle; show that the probability that the time of descent for a particle down the straight line PQ is less than that from P down the straight line of quickest descent to the circle, is $\frac{1}{2}$ 111

6249. (Prof. Matz, M.A.)—Prove that the mean minimum eccentricity of an ellipse capable of resting in equilibrium on a *rough* inclined plane is $e_1 = 2(\sqrt{2} - 1)$ 67

6259. (G. S. Carr, B.A.) — A railway carriage door of width $2a$ and mass m, with the axis through its hinges inclined α to the vertical in a plane at right angles to the rails, hangs open at rest. The carriage suddenly moves forward with an initial velocity v, and a constant acceleration afterwards f. Show

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that—(1) if the door slams, the force of the blow on the jamb is $\frac{1}{3}m[v^2+\frac{8}{3}a(f-g\sin\alpha)]^{\frac{1}{2}}$; and (2) the time of a small oscillation of the door about the position of equilibrium, is

$$4\pi(\tfrac{1}{3}a)^{\frac{1}{2}}(f^2+g^2\sin^2\alpha)^{-\frac{1}{4}}$$ 57

6266. (E. W. Symons, M.A.) — Prove that the tetrahedron of reference, and the tetrahedron formed by the tangent planes at its vertices to the conicoid

$$(l+m+n)(lyz+mzx+nxy) = mn\,xw+nl\,yw+lm\,zw,$$

are such that the lines joining corresponding vertices meet in a point; and that, when l, m, n vary, this point moves on the surface $x^{-1}+y^{-1}+z^{-1}+w^{-1}=0$. 63

6279. (Prof. Townsend, F.R.S.) — An annular lamina of uniform density, bounded internally and externally by concentric circles of any radii, being supposed to attract, according to the law of the inverse cube of the distance, a material particle situated anywhere on the concentric sphere, whose radius is the geometric mean between those of its bounding circles; shew, by any method, that the entire resultant attraction is perpendicular to its plane 73

6281. (Prof. Wolstenholme, M.A.)—Given two real foci and the real asymptote of a circular cubic; prove that the locus of the other real foci is a circle whose centre lies on the given asymptote. 115

6290. (Rev. F. D. Thomson, M.A.)—Find the envelop of the axis of a parabola inscribed in a given triangle 38, 100

6292 & 6325. (J. McDowell, M.A., and R. Knowles, B.A., L.C.P.)—If p_r denote the coefficient of x^r in the expansion of $(1+x)^n$, where n is a positive integer, prove that

$$p_1-\tfrac{1}{2}p_2+\tfrac{1}{3}p_3-\ldots+\frac{1}{n}(-\tfrac{1}{2})^{n-1}p_n = 1+\tfrac{1}{2}+\tfrac{1}{3}+\ldots+\frac{1}{n} \quad\ldots..(1),$$

$$p_1-2p_2+\ldots+(n-1)(-)^{n-2}p_{n-1}=0 \quad\ldots..(2),$$

$$\tfrac{1}{2}p_1-\tfrac{1}{3}p_2+\ldots+\frac{1}{n+1}(-)^{n-1}p_n=\frac{1}{n+1} \quad\ldots..(3).$$ 96

6293. (J. W. Russell, M.A.)—Two bags A and B contain each two balls, each of which is either white or black; a ball is drawn from A and put into B, and then a ball is drawn from B and put into A; in both cases the ball turns out to be white; and this double operation is repeated n times: find the probability of its not failing in the $(n+1)^{\text{th}}$ attempt. 76

6298. (A. Martin, M.A.)—If three equal circles be piled up at random on a horizontal plane, prove that the probability that the pile will stand, is $\frac{1}{16}+\frac{3\sin^{-1}\frac{1}{3}}{8\pi}-\frac{9\sqrt{15}}{256\pi}$..... 43

6307. (Prof. Crofton, F.R.S.)—The vertex A of a triangle is fixed, the base a is a given length taken anywhere along a fixed line; and from a fixed point on this line a parallel to b is drawn, intersecting c in P; find the locus of P. 24

6308 & 6283. (Prof. Francesco Tirelli.) — Se si ha un triangolo ABC, e si congiungono i vertici ad un punto O del cerchio circoscritto al triangolo, e si projetta ciascun vertice sulla congiungente il punto O agli altri due, le tre rette, che uniscono le due projezioni di ciascun vertice, concorrono in un punto della retta, che unisce le projezioni di O su lati del triangolo ABC.

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Se si ha un triangolo ABC ed un punto O del cerchio circoscritto, e si forma il quadrangolo L_1, di cui due vertici opposti sono le projezioni C′ ed A′ di O su BA, BC, mentre gli altri due sono le projezioni α_1 e γ_1 di B su OA, OC; il quadrangolo L_2 di cui due vertici opposti sono le projezioni A′ e B′ di O su CB, CA, mentre gli altri due sono le projezioni β_2 ed α_2 di C su OB, OA; ed il quadrangolo, L_3, di cui due vertici opposti sono le projezioni B′ e C′ di O su AC, AB, mentre gli altri due sono le projezioni γ e β di A su OC, OB, questi quadrangoli hanno ciascuno una coppia di lati opposti paralleli, e le coppie di lati opposti C′A′ ed $\alpha_1\gamma_1$, A′B′ e $\beta_2\alpha_2$, B′C′ e $\gamma\beta$ concorrenti in un punto. 24

6309. (Prof. Townsend, F.R.S.)—A variable conic being supposed to touch in every position the four sides of a fixed quadrilateral circumscribed to a circle; show that its two axes of figure envelope the parabola which touches the three diagonals of the quadrilateral, and has for directrix the right line which bisects their three lengths. 31

6312. (Prof. Cochez.) — Trouver le lieu géométrique du sommet d'une parabole de forme invariable qui le meut en restant constamment tangente à deux droites rectangulaires. 23

6313. (Prof. Genese, M.A.)—Prove that, with the usual notation, the distance from the centre of the inscribed circle of a triangle to the orthocentre is $(2r^2-4R^2\cos A\cos B\cos C)^{\frac{1}{2}}$ 72

6315. (Prof. Matz, M.A.)—The dome of the Capitol at Washington rises m ($=396\frac{1}{2}$ feet), above the level of the ground, and is surmounted by the Statue of Freedom, whose height is n ($=18$ feet); show that an observer whose eye is k ($=5\frac{1}{2}$ feet) above the ground, can get the best view of the Statue when his distance from the axis of the dome is

$$\{(m-h)[(m-h)+n]\}^{\frac{1}{2}} = 399{\cdot}8987372 \text{ feet}. \ldots\ldots 61$$

6319. (W. S. B. Woolhouse, F.R.A.S.)—On a given circle, of radius unity, a corner of a triangle is taken at a given distance (c) from the centre; the other two corners are taken at random on the surface; prove that the average area of all such triangles, in parts of the area of the circle, is

$$\frac{1}{\pi^2}\left(\tfrac{4}{9}+\tfrac{2}{3}c^2-\tfrac{1}{6}c^4+\tfrac{1}{36}c^6\right). \ldots\ldots 42$$

6320. (Elizabeth Blackwood.) — Given that $ax+by$ is less than c and greater than d, and that $cx+dy$ is less than a and greater than b; find the limits of x and y; a, b, c, d being all positive... 54

6321. (W. H. H. Hudson, M.A.)—A particle is placed at the extremity of the vertical minor axis of a smooth ellipse, and is just disturbed; show that, if it quit the ellipse at the end of the latus rectum, the eccentricity must satisfy the equation

$$e^6+5e^4+3e^2=5. \ldots\ldots 26$$

6326. (Rev. F. D. Thomson, M.A.) — A conic cuts a cubic in 6 points; prove that the conics of 5-pointic contact at these points meet the cubic again in 6 points on a conic. 118

6327. (Rev. T. W. Openshaw, M.A.) — Normals are drawn at the ends of a chord parallel to the tangent at the point of an ellipse whose eccentric angle is α; prove that the locus of their intersection is

$$2(ax\sin\alpha+by\cos\alpha)(ax\cos\alpha+by\sin\alpha)=(a^2-b^2)^2\sin 2\alpha\cos^2 2\alpha \quad 27$$

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6331. (J. W. Russell, M.A.)—From the centres α, β, γ of the circles escribed to the triangle ABC, are drawn perpendiculars to the sides produced, so as to form a hexagon $\alpha\gamma'\beta\alpha'\gamma\beta'$ whose opposite sides are parallel; show that the perpendiculars from α', β', γ' on the corresponding sides of the triangle meet in a point..... 42

6335. (W. E. Wright, B.A.)—If ds be an element of an arc of an ellipse whose semi-axes are a, b; and a', b' the semi-axes of the confocal hyperbola through the element; prove that

$$\frac{ds}{da'} = \left(1 + \frac{b^2}{b'^2}\right)^{\frac{1}{2}} \text{.....} \text{.....} \quad 86$$

6337. (J. Young, B.A.)—If the equations

$$ax^2 + bx + c = 0, \quad a_1x^2 + b_1x + c_1 = 0$$

have a common root, form the quadratic whose roots are their other roots. 29

6339. (Prof. Sylvester, F.R.S.)—Understanding by an Algebraical Perimeter of a rectilinear figure, the sum of the sides, each taken with the positive or negative sign; prove that the necessary and sufficient condition of all the curves described by points rigidly attached to the connecting link in a 3-bar linkwork being unicursal is that one of the algebraical perimeters of the quadrilateral (real or impossible) formed by the three links and the line of centres shall be equal to zero. 46

6340. (Prof. Townsend, F.R.S.)—The normal at a variable point on a fixed surface being supposed to intersect in every position a fixed right line in the space of the surface; prove, from the requisite equation of condition, that—

(a) Whatever be the position of the line in the space, the curve locus of the variable point is the intersection with the surface of another of the same order, passing through its several points of intersection with the line, and through the several points at which its tangent planes are perpendicular to the line.

(b) When the line at any of the aforesaid points of intersection is itself a normal to the surface, its point of normal intersection is a crunodal double point of the aforesaid curve, the two tangents at which intersect at right angles, and coincide in direction with the two principal tangents to the surface at the point..... 52

6341. (Prof. Minchin, M.A.)—A rigid body moves about a fixed point O, the motion of the instantaneous axis OI being completely given, as also the angular velocity of the body at each instant about it; find the components of the acceleration of any particle P, in the body along OI, a perpendicular to OI and OP, and a line perpendicular to these two directions..... 66

6342. (Prof. Genese, M.A.)—Prove that the envelop of the directrix of a parabola (1) inscribed in, or described about, a triangle, is a conic; (2) touching two straight lines and passing through a point, or passing through two points and touching a straight line, is a quartic. 119

6347. (Frederick Purser, M.A.)—A variable conic being supposed to touch four fixed right lines in a plane; show that its axes of figure envelope in general a curve of the third class, which breaks up, when the four lines touch a common circle, into the parabola of Question 6309 and the centre of the circle..... 32

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6353. (J. Venn, M.A.)—All x is either y and not z, or z and not y. All wx is either both y and z or neither of them. All xy which is not w is z. No x that is neither y nor w is z. Show that under these conditions there is no x. 51, 74

6356. (C. Leudesdorf, M.A.)—If 7 be divided by 5, given 1 and 2 over; and then successively 21 by 5, giving 4 and 1 over; 14 by 5, giving 2 and 4 over; 42 by 5, giving 8 and 2 over, and so on until the figures recur, the result of placing all the quotients after the 7 and adding a decimal point suitably is $\cdot\dot{7}14285\dot{}$, which is equal to $\frac{5}{7}$. Investigate the general form of the fractions $a : b$ which can be expressed as circulating decimals by dividing b by a in the manner explained with reference to the case of $\frac{5}{7}$. 97

6357. (J. E. A. Steggall, M.A.)—Find in how many ways a given number n may be split up into the sum of 1, 2, 3 ... different numbers. 51

6358. (R. Tucker, M.A.)—If O, P are the equidistant angle-point and the orthocentre of a triangle, and ρ_1, ρ_2, ρ_3 the radii of circles about OPA, OPB, OPC respectively; show (1) that

$$OP^2 (\rho_1^{-2} + \rho_2^{-2} + \rho_3^{-2}) = 8 [1 - \cos(A-B)\cos(B-C)\cos(C-A)];$$

and (2) that, if any angle, as C, = 60°, then four of the points, as A, B, O, P, are concyclic..... 52

6362. (E. B. Elliott, M.A.)—If m lines, straight or curved, are drawn across an enclosed portion of area; prove that, if n_1 be the number of simple internal intersections of these lines with one another or themselves, n_2 the number of triple points, n_3 of quadruple, &c., formed by them, the number of separate areas into which they divide the inclosure is $1 + m + n_1 + 2n_2 + 3n_3 + \ldots$ 120

6365. (H. Murphy.) — If from a variable point on any side of a triangle two lines are drawn making given angles with it; prove that the circle through the points of intersection with the other sides and opposite vertex passes also through a fourth fixed point. 58

6374. (Prof. Townsend, F.R.S.)—A conic being supposed to touch the three sides of a triangle, show that the three pairs of tangents to any confocal conic from the three points of contact are three pairs of common tangents to three circles of corresponding radii having their centres at the vertices of the triangle. 103

6375. (Prof. Crofton, F.R.S.)—If a heavy particle be moving, up or down, on a curve (rough or smooth), show that, if it leaves the curve, the parabola described has contact of the second order with the curve. 49

6377. (Prof. Genese, M.A.) — A quadrilateral linkage is capable in any position of having a circle inscribed in it. If one side be fixed, find the locus of the centre of the inscribed circle..... 102

6378. (Prof. Wolstenholme, M.A.)—Prove that, if $n > 1$,

$$\int_0^1 (1-x)^{n-2}\,dx \int_0^{\frac{1}{2}\pi} \frac{d\theta}{(1-x^2\sin^2\theta)^{\frac{1}{2}n}} = \tfrac{1}{4}\pi \left\{ \frac{\Gamma\frac{1}{2}(n-1)}{\Gamma\frac{1}{2}(n)} \right\}^2 \ldots\ldots \quad 75$$

6382. (Elizabeth Blackwood.)—If x, y, z be each taken at random between a and $-a$; find (1) the chance that $x+y+z$ will be between b and $-b$; and explain (2) any easy experimental method by which the result may be verified. [Taking a and b each 1, Miss Blackwood has found the event to happen 65 times in 100 trials.] 60

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6383. (J. Venn, M.A.)—Are there any inconsistencies or redundancies in the following rules:—(1) The *Financial* Committee shall be chosen from among the *General* Committee; (2) No one shall be a member both of the *General* and *Library* Committees unless he be also on the *Financial* Committee; (3) None of the *Library* Committee shall be on the *Financial*?..... 36

6384. (Dr. Hopkinson, F.R.S.)—A heavy wire of uniform section is carried on a series of supports in the same horizontal plane, L_r being the bending moment at the r^{th} point of support, l_r the distance between the $(r-1)^{th}$ and r^{th} support, and m the mass of the wire per unit length; prove that

$$L_{r-}l_r + 2L_r(l_r + l_{r-1}) - L_{r+1}l_{r+1} = \tfrac{1}{4}mg(l_r^3 + l_{r+1}^3)$$..... 47

6386. (W. S. McCay, M.A.)—If two circles cut orthogonally, prove that (1) an indefinite number of pairs of points can be found on their common diameter such that either point has the same polar to one of the circles as the other point has to the other circle; and (2) the distance between such a pair of points subtends a right angle at an intersection of the circles. 41

6391. (J. J. Walker, M.A.)—If O, A, B, C, D are any five points in space, prove that lines drawn from the middle points of BC, CA, AB respectively parallel to the connectors of D with the middle points of OA, OB, OC, meet in one point E, such that DE passes through, and is bisected by, the centroid of the tetrahedron OABC..... 40

6392. (E. B. Elliott, M.A.)—A_1, A_2, A_3, A_4 are the vertices of a tetrahedron, P is the point whose volume coordinates with regard to it (*i.e.*, the ratios $PA_2A_3A_4 : A_1A_2A_3A_4$, &c.) are x_1, x_2, x_3, x_4, and O is any other point. Prove that, a_{12} being the length of the edge A_1A_2,

$$OP^2 = x_1OA_1^2 + x_2OA_2^2 + x_3OA_3^2 + x_4OA_4^2 - \Sigma(a_{12}^2 x_1 x_2)$$..... 90

6396. (C. Leudesdorf, M.A.) — Show that the square of the tangent from any point P to the circle inscribed in the triangle ABC is $\dfrac{a \cdot AP^2 + b \cdot BP^2 + c \cdot CP^2}{a+b+c} - r(r+2R)$,

r being the radius of this circle, and R that of the circumscribed circle..... 53

6399. (Rev. H. G. Day, M.A.) — If (u, v, w) are the distances of a point from the corners A, B, C of a triangle, and (l, m, n) are constants; find the point within the triangle such that $lu + mv + nw$ shall be a minimum, when a minimum is possible. 59

6400. (J. Hammond, M.A.)—Prove that the surface

$$x^3 + y^3 + z^3 - 3xyz = a^3$$

is one of revolution, and find its axis and the equation of the generating curve (referred to its asymptotes as axes). 118

6401. (R. Knowles, B.A., L.C.P.) — Prove that the locus of the middle points of normal chords to a parabola is the curve

$$(y^2 + 2a^2)^2 - 2axy^2 + 4a^4 = 0$$..... 91

6402. (W. E. Wright, B.A.)—Find the envelop of the line joining the feet of the perpendiculars drawn on the sides of a right-angled triangle from any point in the hypotenuse. 71

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6404. (Sir James Cockle, F.R.S.)—Transform

$$y'' + 2(A\cot x + E\tan x)\,y' + \{\mathfrak{E}^2 - (A-E)^2\}\,y = 0 \ldots (1)$$

into an equation of the form called by Boole binomial..... 45

6407. (Prof. Crofton, F.R.S.)—If, in a triangle, C = 60°, prove that $\frac{1}{a+c} + \frac{1}{b+c} = \frac{3}{a+b+c}$..... 67

6410. (Prof. Genese, M.A.)—Conics are drawn with a common focus, and constant major axis: if their centres lie on a fixed circle, prove that the envelop of the series will be a conic having the same focus..... 48

6414. (Elizabeth Blackwood.)—If x, y, z be each taken at random between 1 and − 1, find the chances (1) that $x+y, y+z, z+x$ are all three positive, (2) that $x+y-xy, y+z-yz, z+x-zx$ are all three positive..... 116

6415. (C. W. Merrifield, F.R.S.)—A regular skeleton dodecahedron is made of 30 equal wires connected 3 by 3 with universal joints. It is then flattened symmetrically, so that two opposite pentagonal faces retain their shape—these two faces being connected with the corners of a regular decagon by 10 of the edges. Show that, in order to make the construction possible, each pentagon much be turned through $\frac{1}{2}\tan^{-1}2$..... 94

6416. (Dr. Hopkinson, F.R.S.)—An endless slightly extensible strap is stretched over two equal pulleys; prove that the maximum couple which the strap can exert on either pulley is

$$\frac{2a(c+\pi a)}{c\coth\frac{1}{2}\mu\pi + \frac{2a}{\mu}}\,T,$$

where a is the radius of either pulley, c the distance of their centres, μ the coefficient of friction, and T the tension with which the strap is put on..... 67

6417. (C. J. Monro, M.A.)—The birds carry away the seed as fast as it falls, but the sower has the start of them by n grains. What is the probability that, after they have carried away x grains, y of those left are of the original n? (See *Origin of Species*, 5th ed., p. 381.)..... 84

6420. (Donald McAlister, D.Sc.)—A regular octohedron is cut by a plane parallel to one of its faces; prove that the perimeter of the section is constant..... 70

6422. (E. B. Elliott, M.A.)—Prove that the envelop of the line

$$x\cos\phi + y\sin\phi = ae^{\phi} + be^{-\phi},$$

where ϕ is a variable parameter, is the evolute of the evolute of the evolute of its own evolute..... 104

6429. (R. Tucker, M.A.)—Given two tangents to a parabola, and the point on the axis through which the chord of contact passes; construct the parabola..... 97

6433. (J. W. McKenzie, B.A.)—Find, if possible, a value of n (other than $n = 24$), so that the sum of the squares of the first n natural numbers may be a square..... 106

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6435. (W. E. Wright, B.A.)—Eliminate α between the equations

$$(x^2\cos\beta+\cos\alpha)\cos\alpha+(x+\sin\alpha)\sin\alpha+x\sin\beta=0,$$

$$(x^2\cos\alpha+\cos\gamma)\cos\gamma+(x+\sin\gamma)\sin\gamma+x\sin\alpha=0 \ldots\ldots 78$$

6438. (Prof. Cayley, F.R.S.)—Find, at any point of a plane curve, the angle between the normal and the line drawn from the point to the centre of the chord parallel and indefinitely near to the tangent at the point; and examine whether a like question applies to a point on a surface and the indicatrix section at such point..... 78

6439. (Prof. Crofton, F.R.S.)—Prove that

$$\int e^{ax^2}dx = e^{ax^2}(x^{-1}+1\,.\,x^{-3}+1\,.\,3x^{-5}+1\,.\,3\,.\,5x^{-7}+\ldots)\ldots\ldots 80$$

6441. (The late Prof. Clifford, F.R.S.)—It is known that, if four lines be given, the circles circumscribing the four triangles so formed meet in a point; and that, if five lines be given, the five points so belonging to their five tetragrams lie on a circle [MIQUEL's Theorem, see *Diary* for 1861, p. 55]. Show that this series of propositions is interminable; so that, if $2n$ lines be given, they determine $2n$ circles that meet in a point; and if $2n+1$ lines be given, they determine $2n+1$ points that lie on a circle. 80

6442. (Prof. Minchin, M.A.)—A body of any shape, with a plane base, rests with this base on a rough horizontal plane; a heavy beam movable round a horizontal axis fixed in the plane rests at a single point against the body, the vertical plane through the beam containing the centre of gravity of the body. Show that limiting equilibrium of the system is impossible unless the normal to the surface of contact of the beam and body makes with the vertical an angle greater than the sum of the angles of friction between the body and the beam and the body and the ground..... 83

6443. (Prof. Tait, M.A.)—Show that, whatever functions of x be represented by y and z, we have always $\dfrac{\int yz\,dx}{\int y\,dx} > \epsilon^{\frac{\int y\log z\,dx}{\int y\,dx}}$, all the integrals being taken between the same limits of x; and all the quantities involved being positive. 86

6444. (Prof. Matz, M.A.)—Three points taken at random, one on each side of a plane triangle, are joined by straight lines; show (1) that the mean area of the triangle thus formed is one-fourth of the area of the triangle; and (2) generalize the problem for any number of points taken in different areas or volumes..... 87

6447. (Christine Ladd.)—The width of a croquet hoop, the thickness of its wires, and the diameter of a ball are given: the ball being in a given position, show how to find the conditions that it may just be possible for it to go through the hoop, (1) straight, (2) by hitting one wire, (3) by hitting both wires; assuming that the angle of incidence is equal to the angle of reflection... 81

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6454. (W. H. H. Hudson, M.A.)—If NQ, CY be the perpendiculars from N, the intersection of the ordinate and directrix, and from C, the vertex, on the tangent at any point P of a catenary, and O be the point where the directrix meets the axis, prove that $QY : OC = \widehat{CP} - ON : PN$. 89

6457. (E. B. Elliott, M.A.)—P is one of the intersections of a lemniscate with the circle on the intercept SS′ between its foci as diameter. Prove that the two curves cut at an angle of 60°, and that the bisector of the angle SPS′ is also the bisector of the angle between the two tangents at P. 120

6460. (J. W. Russell, M.A.) — Show that, as the infinite branch of the curve $r = \frac{2a}{1-\theta^2}$ rolls on its initial tangent, the pole will describe a semicircle 89

6461. (H. McColl, B.A.) — Show that the weakest addition that must be made to the premises A : a, B : b, C : c, &c., to justify the inference Q : q, is the implication $Qq' : Aa' + Bb' + Cc' + \ldots$ 85

6463. (H. L. Orchard, M.A.)—Find the locus of the intersection of normals to the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ at the ends of a chord through the point $(a, 0)$. 105

6468. (A. Martin, M.A.)—An auger hole is made through the centre of a sphere; show that the chance that the volume removed does not exceed one-n^{th} of the sphere is $\left[1-\left(1-\frac{1}{n}\right)^{\frac{3}{2}}\right]^{\frac{1}{3}}$. 119

6479. (Prof. Wolstenholme, M.A.)—Prove that the asymptotic cone of a quadric $u = 0$ is $A\left(\frac{\delta u}{\delta x}\right)^2 + 2F\frac{\delta u}{\delta y}\frac{\delta u}{\delta z} + \&c. = 0$, and the equations of its axes are

$$\frac{A\frac{\delta u}{\delta x} + H\frac{\delta u}{\delta y} + G\frac{\delta u}{\delta z}}{\frac{\delta u}{\delta x}} = \frac{H\frac{\delta u}{\delta x} + B\frac{\delta u}{\delta y} + F\frac{\delta u}{\delta z}}{\frac{\delta u}{\delta y}} = \frac{G\frac{\delta u}{\delta x} + F\frac{\delta u}{\delta y} + C\frac{\delta u}{\delta z}}{\frac{\delta u}{\delta z}},$$

where $A = \frac{\delta^2 u}{\delta y^2}\frac{\delta^2 u}{\delta z^2} - \left(\frac{\delta^2 u}{\delta y \delta z}\right)^2$, $F = \frac{\delta^2 u}{\delta z \delta x}\frac{\delta^2 u}{\delta x \delta y} - \frac{\delta^2 u}{\delta y \delta z}\frac{\delta^2 u}{\delta x^2}$, &c. 82

6480. (Prof. Genese, M.A.) — Given a tangent, a point, and a focus of a conic; prove that the locus of the centre is a conic... 112

6483. (R. Tucker, M.A.)—From P, any point on the line $x = 2a$, tangents PT, PT′ are drawn to a parabola; the normals at T, T′ (as is known) intersect on the curve (in Q). Prove that TT′ passes through a fixed point on the axis, on which axis also lies the centroid of Q, T, T′, the circle about PTQT′ passes through the vertex, and its envelop is a cubic curve; and the orthocentre of QTT′ lies on the diameter through P and on a parabola. 115

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6484. (W. H. Besant, M.A.)—Having given two circles of radii R and r, and that the distance between their centres is $(R^2-2Rr^{\frac{1}{2}})$, prove that an infinite number of triangles can be drawn which shall be inscribed in one circle and circumscribed about the other..... 117

6488. (W. B. Grove, B.A.)—If my son is to enter either the law or the church, he must go either to Oxford or to Cambridge. If he goes to Oxford without entering the law, or to Cambridge without entering the church, he will get a legacy at his uncle's death. He will fail to get the legacy under the following circumstances:—If he will not go to Oxford, and at the same time will not enter the church; or if he will not go to Cambridge, and at the same time will not enter the law—and under *no other* circumstances. I decide that he must enter either the law or the church: will he or will he not get the legacy?..... 80

6489. (T. P. Kirkman, M.A., F.R.S.)—On a square chequer of n^2 small squares, stand n^2-1 counters marked $1, 2, 3 \ldots (n^2-1)$, in any order S, covering all the squares except the last, at the right-hand corner; this last is occupied by a King, visible or invisible. The King, on whatever square he happens to stand, may exchange places only with any counter on a compartment collateral with his. The problem for solution is, to move the King about the board so that, when he has returned to his own final square, the order S shall be reduced to the natural order, the counters marked $a, b, c \ldots$ standing on the $a^{\text{th}}, b^{\text{th}}, c^{\text{th}} \ldots$ squares. Required a rule whereby, on inspection of the order S, the solution can be proved possible or impossible..... 113

6499. (J. L. McKenzie, B.A.)—If x_n and y_n be the n^{th} positive integral solution of $x^2-Ny^2=1$, prove that $x_{n+p}=2x_px_n-x_{n-p}$, and $y_{n+p}=2x_py_n-y_{n-p}$. 114

6503. (Sir James Cockle, F.R.S.)—Transform the equation (1) of Question 6404 into

$$Y''+\left(\varpi\tan t-\frac{i\lambda}{\varpi}\right)Y'+\tfrac{1}{4}\left\{1-\left(\varpi+\frac{\lambda}{\varpi}\right)^2-4N\right\}Y=0\ldots\text{(i.)},$$

where $i=(-1)^{\frac{1}{2}}$; show that (i.) admits of a continuous transformation whereby ϖ is changed into $\varpi-2$, $\varpi-4$, and so on; and notice the case of $\varpi=0$..... 109

MATHEMATICS

FROM

THE EDUCATIONAL TIMES,

WITH ADDITIONAL PAPERS AND SOLUTIONS.

6243. (By Prof. SYLVESTER, F.R.S.)—If $2f^2+g^2$ is a prime and f an odd number, prove that $fy^2+2gxy-2fx^2 = \pm 1$ is soluble in integers.

I. *Solution by* SAMUEL ROBERTS, M.A.

In this case $2f^2+g^2$ is a prime of the form $8m+3$.

Now, if $P^{\frac{1}{2}} = (2f^2+g^2)^{\frac{1}{2}}$ be developed as a continued fraction in the usual way, and $$\frac{P^{\frac{1}{2}}+I}{D}, \quad \frac{P^{\frac{1}{2}}+I_1}{D_1}, \quad \frac{P^{\frac{1}{2}}+I_2}{D_2}$$ are three complete successive quotients in the process, GOEPEL has shown that (1), if $f<2g$, we shall arrive at

$$D = f, \quad I_1 = g, \quad D_1 = 2f, \quad \text{or} \quad D = 2f, \quad I_1 = g, \quad D_1 = f;$$

or (2), if $f>2g$, we shall arrive at

$$D = f+2g, \quad I_1 = f-g, \quad D_1 = f, \quad D_2 = f-2g, \quad I_2 = f+g.$$

In case (1), the forms $(\pm f, g, \mp 2f)$, $(\pm 2f, g \mp f)$, are reduced forms of the principal class. In case (2), the forms

$$[\pm(f+2g), f-g, \mp f], \quad (\pm f, f+g), \quad \mp(f-2g)$$

are reduced forms of the principal class.

But $(f+2g, f-g, -f)$ is properly transformed into $(2f, -g, -f)$ by the substitution $\begin{pmatrix} 1, & 0 \\ 1, & 1 \end{pmatrix}$, and $[f, f+g, -(f-2g)]$ is properly transformed into $(f, g, -2f)$ by the substitution $\begin{pmatrix} 1, & -1 \\ 0, & 1 \end{pmatrix}$. Hence, in both cases, we can make $(f, g, -2f) = 1$, $(2f, -g, -f) = 1$, in integers.

II. *Solution by the* PROPOSER.

Since $x^2-Ay^2 = 1$ is always soluble in integers, we may make

$$2f^2+g^2 = A, \quad u^2-Av^2 = 1,$$

and write $$u+1 = \sigma p^2, \quad u+1 = A\sigma q^2$$

(p, q denoting relative primes).

Hence $$p^2 - Aq^2 = \frac{2}{\sigma} = \pm 1 \text{ or } \pm 2.$$

In fact we must have $$p^2 - Aq^2 = 1 \text{ or } -2,$$

because A being odd, and of the form $2f^2 + g^2$ (where f is odd), is of the form $8n + 3$; and consequently the equations

$$p^2 + 1 = Aq^2 \quad \text{and} \quad p^2 - 2 = Aq^2$$

are each impossible, as is apparent, without recourse to the theory of quadratic residues, by considering each of them with regard to the modulus 8.

If $$p^2 - Aq^2 = 1, \quad v = 2pq,$$

and substituting p, q for u, v, and p_1, q_1 for p, q, we shall obtain in like manner $$p_1^2 - Aq_1^2 = 1 \text{ or } = -2.$$

If $p_1^2 - Aq_1^2 = 1$, we must have $v = 2pq = 4pp_1q$, and continuing this process, since v can only contain 2 a finite number of times, we must come at last to a cessation of the alternative, and to the equation

$$\pi^2 - A\phi^2 = -2, \quad i.e., \quad \pi^2 + 2A = \phi^2,$$

where π, ϕ will both be odd.

Every prime divisor then of $\pi^2 + 2$ we know is of the form $8n + 1$, or $8n + 3$, and therefore is itself of the form $r^2 + 2s^2$, whence, by a known principle of the theory of numbers, it follows that we must have

$$\pi \pm \sqrt{-2} = (g + f\sqrt{-2})\,(g + x\sqrt{-2})^2.$$

Whence, equating the imaginary parts, we find

$$fy^2 + 2gxy - fy^2 = \pm 1.$$

6005. (By the EDITOR.)—If three lengths are taken at random, and the limits be the same (and any magnitude) for all three, show that the respective probabilities of being able to form with them (1) a triangle of any kind, (2) an obtuse-angled triangle, are $p_1 = \frac{1}{2}$, $p_2 = \frac{1}{4}\pi$.

Solution by C. J. MONRO, M.A.

1. It is meant that dx, dy, dz shall be proportional to the chance that the lengths are between x, y, z and $x + dx$, $y + dy$, $z + dz$ respectively. Therefore the chance required is the range of a point having the lengths for rectangular coordinates, as a fraction of a cube with a corner at the origin and three edges along the axes, of a length equal to the common limit. The well-known condition excludes the point from the three tetrahedrons cut off by planes through the origin and adjacent face-diagonals. Having common edges in the faces of the cube, these three have no volume in common; and the volume of each is $\frac{1}{6}$.

2. Cut off quarter-cones having their axes along the coordinate axes and generators along face-diagonals, and you leave $1 - \frac{1}{4}\pi$ for the chance that the triangle is acute-angled, and therefore $\frac{1}{4}\pi$ for the chance of an obtuse-angled triangle. [Another solution of the first part is given, under Quest. 1977, on p. 45 of Vol. IX. of the *Reprint*.]

Note on Solution of Linear Differential Equations with Constant Coefficients. *By* Prof. Wolstenholme, M.A.

In any equation $f\left(\frac{d}{dx}\right) y = X$, where f is a rational integral function, and X a function of x, if one term in X be $\cos nx$ or $\sin nx$, the corresponding term in y is most easily found as follows:—Denoting $\frac{d}{dx}$ by D, let

$$f(D) = \phi_1(D^2) + D\phi_2(D^2), \text{ then } \frac{1}{f(D)} = \frac{\phi_1(D^2) - D\phi_2(D^2)}{(\phi_1 D^2)^2 - D^2(\phi_2 D^2)^2};$$

and therefore $$\frac{1}{f(D)} \cos nx = \frac{\phi_1(-n^2)\cos nx + n\phi_2(-n^2)\sin nx}{[\phi_1(-n^2)]^2 + n^2[\phi_2(-n^2)]^2},$$

and $$\frac{1}{f(D)} \sin nx = \frac{\phi_1(-n^2)\sin nx - n\phi_2(-n^2)\cos nx}{[\phi_1(-n^2)]^2 + n^2[\phi_2(-n^2)]^2}.$$

This will not apply when $f(-n^2) = 0$, but it will then much simplify the work to write $f(D) = (D^2+n^2)^r \phi(D)$, and to operate first with $\frac{1}{\phi(D)}$ in the way just shown, then

$$\frac{1}{(D^2+n^2)^r} \cos nx = \tfrac{1}{2} \frac{x^r}{r!} \left(\frac{e^{nix}}{(2ni)^r} + \frac{e^{-nix}}{(-2ni)^r} \right),$$

and similarly for $\sin nx$.

6312. (By Prof. Cochez.)—Trouver le lieu géométrique du sommet d'une parabole de forme invariable qui le meut en restant constamment tangente à deux droites rectangulaires.

Solution by Prof. Townsend, F.R.S.; Prof. Tanner, M.A.; *and others.*

If, in any position of the parabola, x and y be the coordinates of the vertex to the two rectangular lines as axes, m the modulus of the curve, and θ the angle made by its axis of figure with the axis of x; then since, from the geometry of the parabola, we have $x\cos\theta = m\sin^2\theta$ and $y\sin\theta = m\cos^2\theta$, therefore at once, by elimination of θ, we get

$$x^{\frac{2}{3}} y^{\frac{2}{3}} (x^{\frac{2}{3}} + y^{\frac{2}{3}}) = m^2;$$

which accordingly is the equation, in its most symmetrical form, of the locus in question.

Cleared of radicals, this equation is manifestly equivalent to

$$x^2y^2(x^2 + y^2 + 3m^2) = m^6;$$

from which it appears at once that the locus is of the sixth degree, passes through the two circular points at infinity, and consists of four similar and equal hyperbolical branches, lying asymptotically in the four right angles determined by the axes, touching externally at their four vertices the circle of radius m round the origin as centre, and having all inflexional contact with the axes at their two real intersections with infinity.

6307. (By Prof. CROFTON, F.R.S.)—The vertex A of a triangle is fixed, the base a is a given length taken anywhere along a fixed line; and from a fixed point on this line a parallel to b is drawn, intersecting c in P; find the locus of P.

Solution by C. J. MONRO, M.A.; J. E. A. STEGGALL, B.A.; *and others.*

A′ being the fixed point in BC, complete the parallelograms ABA′C′, ACA′B′: if C may be on either side of B, the locus will be symmetrically related to the fixed points and lines. Join AA′, and draw PM an ordinate to it parallel to the other fixed lines. Then

$$\frac{AM}{AA'} = \frac{AP}{AB} = \frac{CA'}{CB} = \frac{AA'}{MA'} \cdot \frac{MP}{CB}, \text{ or } AM \cdot MA' = \frac{AA'^2}{a} \cdot MP.$$

Thus P lies on either of the parabolas which cut the lines at A, A′, and cut the parallel through the mid-point of AA′ at a distance $\frac{1}{4}a$ from that point. In the parallelogram APA′P′, P and P′ belong to different parabolas.

6241. (By E. W. SYMONS, B.A.)—The roots of the equation

$$x^3 + \lambda x = (x^2 + 1)(2\lambda + \mu)^{\frac{1}{2}}$$

are all real and positive; prove that neither λ nor μ can be less than 9.

Solution by R. TUCKER, M.A.; J. A. KEALY, M.A.; *and others.*

Writing the equation in the form $x^3 + ax^2 + \lambda x + a = 0$, we have, in the case supposed, (see TODHUNTER'S *Theory of Equations*, Art. 173,)

$$4a^4 + a^2(27 - 18\lambda - \lambda^2) + 4\lambda^3 = \text{negative quantity} = U \ldots\ldots\ldots(1).$$

Now, $16U = \{8a^2 - (\lambda^2 + 18\lambda - 27) + [(\lambda - 1)(\lambda - 9)^3]^{\frac{1}{2}}\}$
$\times \{8a^2 - (\lambda^2 + 18\lambda - 27) - [(\lambda - 1)(\lambda - 9)^3]^{\frac{1}{2}}\}$.

Hence, that (1) may be satisfied, ($\because$ λ must be + ve and > 1) $\lambda \not< 9$, and $8a^2$, *i.e.*, $8(2\lambda + \mu) \not< (81 + 162 - 27) \not< 216$, *i.e.*, $2\lambda + \mu \not< 27$, or $\mu \not< 9$.

6308, 6283. (By Prof. FRANCESCO TIRELLI.)—Se si ha un triangolo ABC, e si congiungono i vertici ad un punto O del cerchio circoscritto al triangolo, e si projetta ciascun vertice sulla congiungente il punto O agli altri due, le tre rette, che uniscono le due projezioni di ciascun vertice, concorrono in un punto della retta, che unisce le projezioni di O su lati del triangolo ABC.

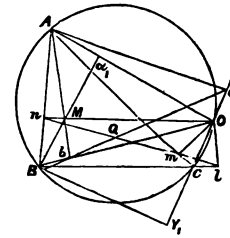
Se si ha un triangolo ABC ed un punto O del cerchio circoscritto, e si forma il quadrangolo L_1, di cui due vertici opposti sono le projezioni C′ ed

A′ di O su BA, BC, mentre gli altri due sono le projezioni α_1 e γ_1 di B su OA, OC; il quadrangolo L_2 di cui due vertici opposti sono le projezioni A′ e B′ di O su CB, CA, mentre gli altri due sono le projezioni β_2 ed α_2 di C su OB, OA; ed il quadrangolo L_3, di cui due vertici opposti sono le projezioni B′ e C′ di O su AC, AB, mentre gli altri due sono le projezioni γ e β di A su OC, OB, questi quadrangoli hanno ciascuno una coppia di lati opposti paralleli, e le coppie di lati opposti C′A′ ed $\alpha_1\gamma_1$, A′B′ e $\beta_2\alpha_2$, B′C′ e $\gamma\beta$ concorrenti in un punto.

Solution by D. Edwardes; Professor Genese, M.A.; *and others.*

(6308.) Let bc' be the pedal line of A with regard to OBC, and lmn that of O with regard to ABC; and let bc', ln meet in Q. Then

$$\angle Qnc' = \angle mAc' \text{ (in circle } AnmOc')$$
$$= \tfrac{1}{2}\pi - \angle ACc' = \tfrac{1}{2}\pi - \angle ACO = \tfrac{1}{2}\pi - \angle ABO;$$
and $$\angle Qc'n = \angle bOn = \tfrac{1}{2}\pi - \angle ABO,$$
therefore $$\angle bQn = \pi - 2\angle ABO.$$



Therefore Q lies on the nine-point circle of ABO, and Q is easily seen to lie on the nine-point circle of ACO. Hence, since the pedal lines A (OBC), O (ABC) intersect on the nine-point circles of ABO and ACO, it follows that the pedal lines A (OBC), B (OAC) intersect on the nine-point circles of ABO and ABC. Hence these four pedal lines meet in a point, through which pass the nine-point circles of the four triangles. [It can be shown that the pedal line O (ABC) bisects the line joining O to the orthocentre of ABC.—(W. H. Besant, in Math. Society's *Proceedings*.) Hence, if H, K, L, M be the orthocentres of ABC, BCD, COA, OAB, then HO, KA, LB, MC meet in a point.]

This also proves the latter part of Question 6283.

For the first part, we have $\angle CB\gamma_1 = \frac{1}{2}\pi - \angle OCl = \frac{1}{2}\pi - \angle A = \angle AOn$. Also $\angle ln\gamma_1 = \angle C\beta\gamma_1$ (in same segment), and $\angle nla_1 = \angle \alpha_1 On$ (in same segment); therefore $\angle ln\gamma_1 = \angle nla_1$, therefore $n\gamma_1$ is parallel to la_1, and similarly for the other quadrilaterals.

6232. (By J. J. Walker, M.A.)—A system of forces being reduced to two, one of which passes through a fixed point and makes a given angle with the resultant of translation; show that the other has for its envelop an ellipse in the plane through the fixed point perpendicular to the axis of principal moment for that point; and that the ellipse becomes a circle when the fixed point is Poinsot's centre.

Solution by W. J. C. Sharp, M.A.; G. H. Hopkins, M.A.; *and others.*

Take OG, the axis of principal moment at the given point O, as axis of z, and the plane through the resultant R as plane of xz; so that $\sin\psi$, 0, $\cos\psi$, are the direction cosines of R; and let P in the direction α, β, γ, and

Q in that of α', β', γ', acting at $(x'y'z')$, be the two forces, and A the given angle between P and R. Then we have

$$\cos\alpha\sin\psi + \cos\gamma\cos\psi = \cos A,\quad R\sin\psi = P\cos\alpha + Q\cos\alpha' \ldots(1, 2),$$

$$R\cos\psi = P\cos\gamma + Q\cos\gamma',\quad 0 = P\cos\beta + Q\cos\beta' \quad\ldots\ldots(3, 4),$$

$$0 = Q(y'\cos\gamma' - z'\cos\beta'),\quad 0 = Q(z'\cos\alpha' - x'\cos\gamma') \quad\ldots\ldots(5, 6),$$

$$G = Q(x'\cos\beta' - y'\cos\alpha') \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (7).$$

From (5) and (6), $\frac{x'}{\cos\alpha'} = \frac{y'}{\cos\beta'} = \frac{z'}{\cos\gamma'}$, which imply a single resultant, or $z' = 0$, $\cos\gamma' = 0$; *i.e.*, Q lies in the plane of xy, and the equation to its line of action is $x\cos\beta' - y\cos\alpha' = \frac{G}{Q}$. Now, from (1), (2), and (3),

$$R = P\cos A + Q\cos\alpha'\sin\psi,$$

and from (2), (3), and (4), $R^2 - 2RQ\cos\alpha'\sin\psi + Q^2 = P^2$.

Therefore $Q^2(1-\cos^2\alpha'\sin^2\psi) = P^2\sin^2 A$; therefore the equation to Q is $y + \frac{G\sin\psi}{R} = x\tan\alpha' - \frac{G\cot\alpha}{R}(\tan^2\alpha' + \cos^2\psi)^{\frac{1}{2}}$, the envelop of which is the ellipse $x^2 + \left(y + \frac{G\sin\psi}{R}\right)^2 \frac{1}{\cos^2\psi} = \frac{G^2}{P^2\sin^2 A}$, which becomes a circle when $\psi = 0$, that is to say, when the point is Poinsot's centre.

6321. (W. H. H. Hudson, M.A.)—A particle is placed at the extremity of the vertical minor axis of a smooth ellipse, and is just disturbed; show that, if it quit the ellipse at the end of the latus rectum, the eccentricity must satisfy the equation $e^6 + 5e^4 + 3e^2 = 5$.

Solution by Prof. Scott, M.A.; D. Edwardes; *and others.*

If v be the velocity at the one end of the minor axis, $v^2 = 2g\left(b - \frac{b^2}{a^2}\right)$; and the chord of curvature at the extremity of the *latus rectum*, parallel to the minor axis, is $2a(1-e^4)$; therefore

$$v^2 = 2g \cdot \tfrac{1}{4} \cdot 2a(1-e^4);\ \text{hence}\ 2\left(b - \frac{b^2}{a}\right) = a(1-e^4),$$

or $$2 = (1-e^2)^{\frac{1}{2}}(3+e^2),\ \text{whence}\ e^6 + 5e^4 + 3e^2 = 5.$$

6000. (By W. S. B. Woolhouse, F.R.A.S.)—The separate probabilities of three partially dependent events are given as well as the probabilities of the joint occurrence of every two events and also the joint occurrence of all three. In a proposed case, however, it is known that two of the

events have failed, but it is unknown which; determine the probability of the failure also of the remaining event.

Solution by C. J. MONRO, M.A.

In BOOLE's forms let x, y, z be the events, and let a, b, c, l, m, n, t be the chances of x, y, z, yz, zx, xy, xyz.

These given, so are the relative frequencies of the known circumstance

$$x(1-y)(1-z)+y(1-x)(1-z)+z(1-x)(1-y)+(1-x)(1-y)(1-z) = 1-yz-zx-xy+2xyz,$$

and of the circumstance in question $(1-x)(1-y)(1-z)$. Therefore the chances of this is given as a ratio between the two; namely,

$$\frac{1-a-b-c+l+m+n-t}{1-l-m-n+2t}.$$

6327. (By the Rev. T. W. OPENSHAW, M.A.)—Normals are drawn at the ends of a chord parallel to the tangent at the point of an ellipse whose eccentric angle is α; prove that the locus of their intersection is

$$2(ax\sin\alpha+by\cos\alpha)(ax\cos\alpha+by\sin\alpha) = (a^2-b^2)^2\sin 2\alpha\cos^2 2\alpha.$$

Solution by J. E. A. STEGGALL, B.A.; R. L. KNOWLES, L.C.P.; *and others.*

The eccentric angles of the other points are $(\alpha+\theta, \alpha-\theta)$; hence the normals are $\dfrac{ax}{\cos(\alpha\pm\theta)} - \dfrac{by}{\sin(\alpha\pm\theta)} = a^2-b^2$;

therefore $(ax\sin\alpha - by\cos\alpha)\cos\theta = (a^2-b^2)\sin 2\alpha\cos 2\theta$ .....(1),

$(ax\cos\alpha+by\sin\alpha)\sin\theta = (a^2-b^2)\cos 2\alpha\sin 2\theta$ .....(2).

Multiply by $\cos 2\alpha\sin\theta$ and $\sin 2\alpha\cos\theta$, and subtract,

$\cos\theta\sin\theta\,(ax\sin\alpha+by\cos\alpha)$ .....(3).

Multiply (2) and (3), and we get the result in the Question.

6019, 6088. (By T. R. TERRY, M.A.)—If $a<1$ and m and n are zero, or any positive integers, and

$$f(n,m) = \frac{(2n)!}{2^{2n}n!\,n!}\left\{1-\frac{n-m}{n+1}ma^2+\frac{(n-m)(n-m+1)}{n+1\,.\,n+2}\cdot\frac{m(m-1)}{1\,.\,2}a^4-\ldots\right\},$$

prove that $4a^2f(n,m) = -f(n-1,m+1)+2(1+a^2)f(n-1,m)$

$-(1-a^2)^2f(n-1,m-1)$.....(1),

$$\frac{df(n,m)}{da} = 4a\frac{(n-m)(n-m+1)}{2n+1}f(n+1,\ m-1) - 2a(n-m)f(n,m-1)\ldots(2),$$

$$f(n,m+1) = (1+a^2)f(n,m) - \frac{4a(n-m)}{2n+1}f(n+1,m)\ \ldots\ldots\ldots(3).$$

Solution by the PROPOSER; J. HAMMOND, M.A.; *and others.*

It may be shown that, if $\Delta = 1 - 2a\cos x + a^2$,

$$\int_0^\pi \frac{\sin^{2n}x\,dx}{\Delta^{n-m}} = \pi f(n,m)\ldots\ldots\ldots\ldots\ldots\ldots(a).$$

Hence (1) follows from the obvious truth that

$$\frac{\sin^2 x}{\Delta^2} = -\frac{1}{4a^2} + \frac{1+a^2}{2a^2}\frac{1}{\Delta} - \frac{(1-a^2)^2}{4a^2}\frac{1}{\Delta^2}.$$

Differentiating (a) with respect to a, we get (2); and, integrating $\int_0^\pi \frac{\sin^{2n}x\cos x\,dx}{\Delta^{n-m}}$ by parts, we get (3).

[The restriction on m is unnecessary. The results may also be obtained from the known properties of the hypergeometric series. See also "Notes on a Class of Definite Integrals," by the Proposer, in Vol. XI. of the *Proceedings* of the Mathematical Society.]

5973 & 6002. (By J. L. MACKENZIE, B.A.) — Solve the functional equation $\phi(ax+\beta) = \gamma\phi(x) + u$, where a, β, γ are constants and u is a known function of x; and, as particular cases, solve the equations

$$\phi(a-x)\phi(x) + b[\phi(a-x) + \phi(x)] = c^2, \quad \phi(mx) = \phi(x) + ax\log x.$$

Solution by J. HAMMOND, M.A.; H. STABENOW, M.A.; *and others.*

Let P denote an operator such that $P\phi(x) = \phi(ax+\beta)$; C a constant of solution, that is, a function unaffected by the operation P; and S a particular solution of the equation $P = \gamma$, that is, a function such that $PS = \gamma S$; then the solution of the general form is

$\phi(x) = CS + (P-\gamma)^{-1}u$, for $P(CS) = PC \cdot PS = \gamma CS$, and $(P-\gamma)(CS) = 0$.

(Qu. 5973.) Adding b^2 to each side and taking logarithms, we have

$$\log[\phi(a-x)+b] + \log[\phi(x)+b] = \log(b^2+c^2);$$

or $\quad (P+1)\log[\phi(x)+b] = \log(b^2+c^2) \quad [a=-1,\ \beta=a,\ \gamma=-1].$

And the solution is $\log[\phi(x)+b] = CS + \frac{1}{P+1}\log(b^2+c^2)$

$$= CS + \tfrac{1}{2}\log(b^2+c^2) \text{ [since } P=1 \text{ for constants]}.$$

Here C, S are arbitrary functions $F(x)$, $f(x)$, such that

$$F(x) = F(a-x),\ \ f(x) = -f(a-x);$$

and we may without loss of generality take the particular solution $S = \frac{1}{2}a - x$, C being in this case any even function of $(\frac{1}{2}a-x)$.

When $S = \frac{1}{2}a - x$ and $C = -a^{-1}\log(b^2+c^2)$, we have

$$\log[\phi(x)+b] = \frac{x}{a}\log(b^2+c^2), \text{ or } \phi(x)+b = (b^2+c^2)^{\frac{x}{a}}.$$

When $S = \log\tan\frac{\pi x}{2a}$, and $C = 1$, $\phi(x)+b = (b^2+c^2)^{\frac{1}{2}}\tan\frac{\pi x}{2a}$.

Both of these cases are included in the general solution

$$\phi(x) = (b^2+c^2)^{\frac{1}{2}}e^{CS} - b.$$

(Qu. 6002.) Here $\phi(x) = C + \frac{1}{P-1}ax\log x$ $[\alpha = m, \beta = 0, \gamma = 1]$.

Now $P^n ax\log x = am^n x\log(m^n x) = am^n x\log x + a\mu m^n x\log m$.

Thus $F(P)\,ax\log x = ax\log x\,F(m) + am\,x\log m\,F'(m)$,

and the solution becomes

$$\phi(x) = C + \frac{ax\log x}{m-1} - \frac{amx\log m}{(m-1)^2},$$

where C is unaltered by the change of x into mx.

6337. (By J. Young, B.A.)—If the equations

$$ax^2+bx+c = 0, \quad a_1x^2+b_1x+c_1 = 0$$

have a common root, form the quadratic whose roots are their other roots.

Solution by G. Heppel, M.A.; Prof. Genese, M.A.; *and others.*

Let $(-\alpha, -\beta)$ be the roots of the first, $(-\alpha, -\gamma)$ of the second; then

$$\alpha+\beta = \frac{b}{a}, \quad \alpha\beta = \frac{c}{a}, \quad \alpha+\gamma = \frac{b_1}{a_1}, \quad \alpha\gamma = \frac{c_1}{a_1};$$

whence

$$\alpha = \frac{ac_1-a_1c}{ab_1-a_1b} = \frac{bc_1-b_1c}{ac_1-a_1c},$$

therefore

$$(ac_1-a_1c)^2 = (ab_1-a_1b)(bc_1-b_1c) \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (A).$$

Since $\beta = \frac{ab_1-a_1b}{ac_1-a_1c}\cdot\frac{c}{a}$, $\gamma = \frac{ab_1-a_1b}{ac_1-a_1}\cdot\frac{c_1}{a_1}$, the required equation is

$$x^2 + \frac{ab_1-a_1b}{ac_1-a_1c}\left(\frac{c}{a}+\frac{c_1}{a_1}\right)x + \left(\frac{ab_1-a_1b}{ac_1-a_1c}\right)^2\cdot\frac{cc_1}{aa_1},$$

or, by (A), $aa_1(bc_1-b_1c)x^2 + (a^2c_1^2-a_1^2c^2)x + cc_1(ab_1-a_1b) = 0$.

6205. (By W. J. C. Sharp, M.A.)—Prove that, according as the polar conic of a point on a cubic is an hyperbola, a parabola, or an ellipse, two lines, one line, or no line, can be drawn through the point, so as to be terminated by the curve and bisected at the point.

3:1

Solution by J. J. WALKER, M.A.

According to my method of treating the intersections of a transversal, $lx + my + nz = 0$, with the curve $u = 0$ [*Proceedings of Math. Soc.*, Vol. IX., p. 227 (4)] the condition for the bisection of the chord $lx \ldots$ at the point (xyz) on u is (1) $a\lambda^2 + \ldots + 2h\lambda\mu = 0$, where $\lambda = m \sin C - n \sin B, \ldots$, and consequently (2) $\lambda \sin A + \mu \sin B + \nu \sin C \equiv 0$, $a \ldots h$ standing for the second differential coefficients of u, so that the polar conic is $ax^2 + \ldots 2hxy = 0$. Eliminating λ between (1) and (2), the discriminant of the equation for the two values of $\mu : \nu$ is, as is well known, $(-bc + f^2) \sin^2 A + \ldots$, which is negative, zero, or positive, as the polar conic is a hyperbola, parabola, or ellipse.

5791. (By W. H. H. HUDSON, M.A.)—Let AB, BC, CD be equal arcs of a circle, TA, TD be the tangents at A, D; AB, DC meet in Y; XB, XC, the tangents at B, C, meet AD produced in E, F: show that, when B, C, D, move up to A, the ultimate ratio of the triangles TAD, XEF, YAD, are as 27 : 25 : 18.

Solution by the Rev. J. L. KITCHIN, M.A.; NARENDRA LAL; *and others.*

Let AOK (= 2) be a diameter of the circle; and let AB, BC, CD subtend at the centre O an angle θ; then $\Delta YAD = \sin\theta \sin^2 \frac{3}{2}\theta$,

$$\Delta TAD = \sin^3 \tfrac{3}{2}\theta,$$

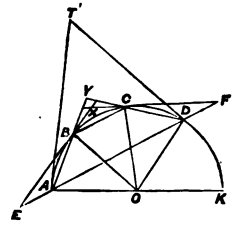
$$\Delta XEF = \sin \tfrac{1}{2}\theta \,(2 \sin \tfrac{1}{2}\theta + \sin \tfrac{3}{2}\theta)^2;$$

therefore $\Delta YAD : \Delta TAD : \Delta XEF$

$$= \sin\theta \sin^2 \tfrac{3}{2}\theta : \sin^3 \tfrac{3}{2}\theta : \sin \tfrac{1}{2}\theta \,(2 \sin \tfrac{1}{2}\theta + \sin \tfrac{3}{2}\theta)^2$$

$$= 1 : \frac{\sin \frac{3}{2}\theta}{\sin\theta} : \frac{\sin \frac{1}{2}\theta}{\sin\theta} \left(\frac{2 \sin \frac{1}{2}\theta}{\sin \frac{3}{2}\theta} + 1 \right)^2$$

= (in limit) $1 : \frac{3}{4} : \frac{1}{4}(\frac{3}{3} + 1)^2 = 18 : 27 : 35.$



6246. (By Prof. TOWNSEND, F.R.S.)—In the rotation, round its axis of figure, of a liquid spheroid of uniform density, held in permanent form by its own attraction for the ordinary law of the inverse square of the distance; show, from the equation of permanence, that

$$\frac{\omega^2}{4\pi\rho} = \frac{2}{3 \times 5}\lambda^2 - \frac{4}{5 \times 7}\lambda^4 + \frac{6}{7 \times 9}\lambda^6 - \frac{8}{9 \times 11}\lambda^8 + \&c.,$$

the several quantities involved having their usual significations.

Solution by the PROPOSER; W. J. C. SHARP, M.A.; *and others.*

The equation of permanence of form of the revolving fluid being, as is well known, $\dfrac{3\lambda + 2\epsilon\lambda^3}{3 + \lambda^2} - \tan^{-1}\lambda = 0$, where $\epsilon = \dfrac{\omega^2}{4\pi\rho}$,

the above follows from it immediately on the substitution for $\tan^{-1}\lambda$ of its development in ascending powers of λ. When λ is so small that all powers of it above the lowest may be neglected in the above series, then approximately $\lambda^2 = \frac{15}{8}\frac{\omega^2}{\pi\rho}$, its well-known approximate value in that case.

6309. (By Prof. TOWNSEND, F.R.S.) — A variable conic being supposed to touch in every position the four sides of a fixed quadrilateral circumscribed to a circle; show that its two axes of figure envelope the parabola which touches the three diagonals of the quadrilateral, and has for directrix the right line which bisects their three lengths.

I. *Solution by* C. TAYLOR, M.A.; Prof. GENESE, M.A.; *and others.*

Call the straight line which bisects the three diagonals L, M, N of the quadrilateral *the diameter of the quadrilateral.* This line is the centre-locus of all conics inscribed in the quadrilateral.

If $C\infty$ and $C\infty'$ be the axes of any one of the inscribed conics, the sides of the triangles $C\infty\infty'$ and LMN, which are self-polar to this conic, touch one *conic*, in this case necessarily a *parabola* as touching the line at infinity. The directrix of this parabola passes through C and through the centre of the circle (which is the orthocentre of the triangle LMN), and therefore coincides with the diameter of the quadrilateral. Hence the parabola is the same for all positions of C, and is that of the question.

II. *Solution by the* PROPOSER.

This property is an obvious consequence from the several following, all of which are well known, viz.:—(1) That the centres of figure of the entire system of conics inscribed to any quadrilateral in a plane lie all on the right line which bisects the three diagonals of the quadrilateral. (2) That the triangle determined by the three diagonals is self-reciprocal with respect to every conic of the system. (3) That, when (as in the question) a conic of the system is a circle, its centre of figure coincides in position with the orthocentre of the triangle, and lies consequently on the directrix of every parabola which touches its three sides. (4) That the two axes of figure of every conic to which a triangle in the same plane is self-reciprocal, are the two rectangular conjugate rays of the involution determined at its centre by the three connectors with the vertices, and by the three parallels to the sides of the triangle. And (5) That the two tangents, from any point in the plane of the triangle, to the inscribed parabola whose directrix passes through the point, are also the two rectangular conjugate rays of the same involution for the point. From these several properties combined, that of the question is an obvious and immediate consequence.

[From (3) and (4) above, it appears at once, as is often useful, that, a triangle being supposed self-reciprocal with respect to a conic in its plane, the two axes of figure of the conic are the two tangents to the parabola inscribed to the triangle whose directrix passes through the centre of the conic.]

6347. (By FREDERICK PURSER, M.A.)—A variable conic being supposed to touch four fixed right lines in a plane; show that its axes of figure envelop in general a curve of the third class, which breaks up, when the four lines touch a common circle, into the parabola of Question 6309 and the centre of the circle.

I. *Solution by* Dr. F. TIRELLI; F. D. THOMSON, M.A.; *and others.*

Il est facile de voir, que les axes des coniques inscrites dans un quadrilatère *abcd* déterminent sur la ligne droite à l'infini de leur plan une involution du deuxième ordre projectife à la ponctuée r des centres des mêmes coniques. Or, il est bien clair, que les joignantes les points d'une ponctuée aux extrêmes segments correspondants d'une involution de deuxième ordre projectife à elle-même enveloppent une courbe de troisième classe. Si l'on suppose cependant, que le quadrilatère *abcd* soit circonscrit à un cercle ayant le centre C, le faisceau des lignes droites perpendiculaires de centre C est un enveloppement de première classe, qui fait partie de l'enveloppement susdit de troisième classe; c'est que, en tel cas, l'enveloppement des axes des coniques inscrites dans le quadrilatère est une conique, qui, à cause des coniques exceptionnelles du système (coniques réduites à deux points) touche aux diagonales du quadrilatère, et dont le cercle directeur se réduit à la ligne droite r. C'est donc pourquoi cette conique est une parabole, qui a r pour directrice.

II. *Solution by the* PROPOSER.

More generally, the axes of a variable conic touching any four fixed lines in a plane envelop a line of the third class, which, when the four lines touch a circle, breaks up into the parabola of the question, and the centre of the circle. This may be easily shown as follows:—

If we denote, for convenience, by the terms "harmonic line" of a given line, the line which is the locus of its poles with respect to a system of conics touching four fixed lines, it is evident that we may define an axis of a conic of the system as a line perpendicular to its harmonic line. For its pole with regard to its own conic is the point at ∞ in the directions perpendicular to it. Let now the equation of a conic of the system referred to the three diagonals of the quadrilateral be $a\alpha^2+b\beta^2+c\gamma^2=0$. Let also $p\alpha+q\beta+r\gamma=0$ be the equation of the line which is the locus of centres of conics of the system, *i.e.*, the harmonic of the line ∞. Then the coordinates of the centre being $\frac{\sin A}{a}, \frac{\sin B}{b}, \frac{\sin C}{c}$, a, b, c will be connected by the relations $\frac{p\sin A}{a}+\frac{q\sin B}{b}+\frac{r\sin C}{c}=0$.

Now, in general the pole of any line $l\alpha+m\beta+n\gamma=0$ is $\frac{l}{a}, \frac{m}{b}, \frac{n}{c}$.

The locus of such poles as the harmonic line is then

$$\frac{p\sin A}{l}\alpha+\frac{q\sin B}{m}\beta+\frac{r\sin C}{n}\gamma=0.$$

The condition that this line should be perpendicular to $l\alpha + m\beta + n\gamma = 0$, is $$p \sin A + q \sin B + r \sin C - \Sigma \left(\frac{qn \sin B}{m} + \frac{rm \sin C}{n} \right) \cos A = 0,$$ or $$\Sigma l \cos A (rm^2 \sin C + qn^2 \sin B) - lmn (p \sin A + q \sin B + r \sin C) = 0.$$ The envelop of a line perpendicular to its harmonic is then, in general, a curve of the third class.

Now, we have, identically,

$$\Sigma l \cos A (rm^2 \sin C + qn^2 \sin B) - lmn (p \sin A + q \sin B + r \sin C)$$
$$= (l \cos B \cos C + m \cos C \cos A + n \cos A \cos B)(mnp \tan A + nlq \tan B + lmr \tan C)$$
$$- \frac{lmn}{\sin A \sin B \sin C} \left(\frac{p}{\cos A} + \frac{q}{\cos B} + \frac{r}{\cos C} \right).$$

If, now, the quadrilateral be circumscribable to a circle, the centre of this circle will be the orthocentre of the triangle of reference.

Expressing, then, that the centre of the circle lies on the general locus of centres, we have $$\frac{p}{\cos A} + \frac{q}{\cos B} + \frac{r}{\cos C} = 0.$$

The general curve of the third class above breaks up then in this case into two factors, one being the point equation of the orthocentre, the other the tangential equation of a conic touching the sides of the triangle of reference, and which in virtue of the equation

$$\frac{p}{\cos A} + \frac{q}{\cos B} + \frac{r}{\cos C} = 0,$$

is touched also by $\alpha \sin A + \beta \sin B + \gamma \sin C = 0$, *i. e.*, is a parabola.

It is easy to see the geometrical reason for the resolution of the general curve into factors which takes place in this case. In fact, when one of the conics of the system is a circle, any line through its centre is an axis of the circle, and therefore a line of the envelop, which must therefore consist of the centre and a conic.

[The general result given above is in accordance with that of Quest. 6290, (whereof a solution is given on p. 34 of this volume,) which is a particular case of it. Here one of the four fixed lines is the line ∞. The curve of the third class is the tricusp which is the envelop of the finite axis; the line at ∞, which here constitutes the other axis of conics of the system, touching the envelop, as it should do, being, in fact, the bitangent to it at the point I, J.]

6169. (By Elizabeth Blackwood.)—If P, Q, R are random points within a sphere of which O is the centre; find the average volume of the tetrahedron OPQR.

Solution by Professor Seitz, M.A.

Let $OP = x$, $OQ = y$, $\angle POQ = \theta$, and let z = the perpendicular distance of R from OPQ, and r = the radius of the sphere. Then $\frac{1}{6} xyz \sin \theta$ = the volume of the tetrahedron; an element of the sphere at P is $4\pi x^2 dx$, at Q

it is $2\pi y^2 \sin\theta\, dy\, d\theta$, and at R it is $\pi(r^2-z^2)\,dz$. The limits of each of the variables x, y, z are 0 and r, and those of θ are 0 and π.

Hence the average volume of the tetrahedron is

$$\frac{\int_0^\pi\int_0^r\int_0^r\int_0^r \tfrac{1}{8}xyz\sin\theta\,.\,8\pi^3\sin\theta\,d\theta x^2\,dx\,y^2\,dy\,(r^2-z^2)\,dz}{\int_0^\pi\int_0^r\int_0^r\int_0^r 8\pi^3\sin\theta\,d\theta\,x^2\,dx\,y^2 dy\,(r^2-z^2)\,dz}$$

$$= \frac{9}{8r}\int_0^\pi\int_0^r\int_0^r\int_0^r \sin^2\theta\,d\theta\,x^3\,dx\,y^3\,dy\,(r^2-z^2)\,z\,dz$$

$$= \frac{9}{32r^5}\int_0^\pi\int_0^r\int_0^r \sin^2\theta\,d\theta\,x^3\,dx\,y^3\,dy$$

$$= \frac{9}{128r}\int_0^\pi\int_0^r \sin^2\theta\,d\theta\,x^3\,dx = \frac{9r^3}{512}\int_0^\pi \sin^2\theta\,d\theta = \frac{9\pi r^3}{1024}.$$

[Mr. MONRO, in his Solution of this Quest. (*Reprint*, XXXIII., 75), "errs," says Prof. SEITZ, "in limiting P to a semi-diameter, and Q to a semi-circle." In reference to this correction, Mr. MONRO writes as follows:—"I certainly erred, as Mr. SEITZ says. The ranges of P and Q should not have been a line and surface at all, but a volume in each case, namely, the infinitesimal cone and wedge of which the semi-diameter and semi-circle will give the finite dimensions. The distances of the mean positions will thus be $\frac{3}{4}$, $\frac{3}{16}\pi$, $\frac{3}{8}$ of the radius; and $\frac{1}{8}$ of the product is $\frac{9}{1024}\pi$.]

5901. (By Prof. SYLVESTER, F.R.S.)—Let μ points be given on a cubic curve. Through them draw any curve (simple or compound) of degree ν; the remaining $3\nu-\mu$ (say μ') points may be termed a first residuum to the given ones. Through these μ' points draw any curve of degree ν'; the remaining $3\nu'-\mu'$ points may be termed a residuum of the second order to the given ones; and in this way we may form at pleasure a series of residua of the third, fourth, and of any higher order. If μ is of the form $3i-1$, a residuum of the first or any odd order; and if μ is of the form $3i+1$, a residuum of the second or any even order in such series, may be made to consist of a single point, which I call *residual* of the original μ points. Prove that any such residual is dependent wholly and solely on the original μ points, being independent of the number, degrees, and forms of the successive auxiliary curves employed to arrive at it.

6094. (By Prof. SYLVESTER, F.R.S.)—1. If the process of residuation, described in Question 2391, be applied to a quartic instead of to a cubic curve, prove that the minimum number of points in any *residuum* is 3, that being the number applicable to the case of an odd number of initial points. If the initial number is of the form $4i$, show that the minimum number in a residuum is 4, and when the initial number is of the form $4i+2$, 6, except for the case when $i=0$, when it is obviously only 2.

2. Prove that, if the number of initial points is odd, the 3 *residuals* are functions only of those points.

3. In general, if p be prime to μ and p initial points be taken in a curve of the μ^{th} order, prove that the residuum may be reduced to consist of $\frac{1}{2}(\mu-1)(\mu-2)$ residual points, which residuals are all functions of the p given points only, being independent of the particular process of residuation applied to arrive at them.

Solution by W. J. Curran Sharp, M.A.

The principle of residuation, which is enunciated in these two theorems, depends upon the proposition that "every curve of the n^{th} degree, which is drawn through $np - \frac{1}{2}(p-1)(p-2)$ points on a curve of the p^{th} degree (p less than n), meets this curve in $\frac{1}{2}(p-1)(p-2)$ other fixed points." [See Salmon's *Higher Plane Curves*, p. 18.]

If $p = 3$, there is only one additional point.

First, if $\mu = 3i-1$, U_i an i-ic will cut in one additional point A.

The ν-ic U_ν will cut in the μ points, and μ_1 other points,
ν_1-ic U_{ν_1} ,, μ_1 ,, μ_2 ,,
&c. &c. &c. &c.
ν_{2m}-ic $U_{\nu_{2m}}$,, μ_{2m}, and one point B,

and $i + \nu_1 + \ldots + \nu_{2m-1} = \nu + \nu_2 + \ldots + \nu_{2m} \equiv r$ suppose,

and the r-ic $U_i U_{\nu_1} \ldots U_{\nu_{2m-1}}$ will intersect the cubic in the points μ, μ_1, &c., and A; and the r-ic $U_\nu . U_{\nu_2} \ldots U_{\nu_{2m}}$ will intersect it in the same $3r-1$ points (the μ, μ_1, &c. points) and B; hence A and B are identical; and similarly if $\mu = 3i+1$.

If $p = 4$, the number of additional points is 3, a value which can only be obtained if μ be odd; since $\mu_1 = 4\nu - \mu$, $\mu_2 = 4\nu_1 - 4\nu + \mu$, &c. Hence also, if $\mu = 4i$, all the residua are of the same form, and if $\mu = i+2$, $\mu_1 > 2$ unless $\nu < i+2$.

If the number of points μ be odd, so that a residuum of 3 is possible, it can be proved exactly as in the case of the cubic that any two residua consisting of three points are identically the same. And so also in the general case.

[Like the corresponding proposition for cubics, the extension in Quest. 6094 includes a vast number of geometrical properties. In No. 1 of Vol. 3 (1880) of the *American Journal of Mathematics* will be found an application of the Theory of Residuation to the determination of all the points on a Cubic Curve rationally connected with any given point. These points, which usually are *infinite*, become for particular positions of the initial point periodic, and consequently *finite* in number. Such positions of the initial point may be termed *sub-inflexions*. Any such sub-inflexion is a point one of whose rational derivatives is a point of inflexion on the conic. The simplest instance of sub-inflexions is afforded by the 27 points of Plücker, viz., the anti-tangentials of the 9 points of inflexion.]

6383. (By J. Venn, M.A.)—Are there any inconsistencies or redundancies in the following rules:—(1) The *Financial* Committee shall be chosen from among the *General* Committee; (2) No one shall be a member

both of the *General* and *Library* Committees unless he be also on the *Financial* Committee; (3) None of the *Library* Committee shall be on the *Financial*?

I. *Solution by the* Rev. R. Harley, F.R.S.; Prof. Matz, M.A.; *and others.*

The given rules contain no inconsistencies, but they may be simplified and reduced by striking out the third rule and the last clause of the second. This becomes obvious on writing them in the Boolian forms.

Let F, G, L stand for member of the *Financial*, *General*, and *Library* Committees respectively, then we have

$$F = FG, \quad GL = FGL, \quad FL = 0 \ldots\ldots\ldots\ldots\ldots (1, 2, 3).$$

Now, (1) and (2) give $GL = FL$, and therefore, by (3), $GL = 0$.....(4). But this is given by (2) and (3); thus, $FGL^2 = FGL \cdot 0$; or $FGL = 0$, and therefore, by (2), $GL = 0$.

Replace then the second and third rules by the following:—

(4) None of the *Library* Committee shall be on the *General*.

Rules (1) and (4) are exactly equivalent in force to the three given rules (1), (2), (3). For we have seen that (2) and (3) give (4), and it may readily be shown that (1) and (4) give (1), (2), and (3); thus,

$$FGL = FG \cdot 0 = 0 = GL,$$

which is (2), and, by (4), $L = \frac{0}{G} = \frac{0}{0}(1-G)$;

therefore, by (1), $FL = \frac{0}{0} FG(1-G) = 0$, which is identical with (3).

II. *Solution by* H. McColl, B.A.; Prof. Genese, M.A.; *and others.*

Speaking of any originally unclassed person, let f, g, l respectively denote the statements: He shall be on the *Financial* Committee; He shall be on the *General* Committee; He shall be on the *Library* Committee. The three rules will then be expressed by the compound implication $(f : g)(gl : f)(l : f')$, which is equivalent to the implication $fg' + glf' + lf : 0$. Reducing the antecedent of this zero implication to its primitive form, we get $fg' + gl : 0$, which is equivalent to $(f : g)(g : l)$, and may be read: Everyone on the Financial Committee shall be on the General Committee, and no one on the General Committee shall be on the Library Committee.

6075. (By Donald McAlister, D.Sc. — A sphere full of water, is hung up by a string. Suddenly a slender uniform crack extends from one end of the vertical diameter to the other. Prove that the time of emptying (on the hypothesis of "parallel sections") is to the time of free fall from rest down the diameter, as the area of the sphere is to that of the lune.

Solution by J. J. Walker, M.A.

The same diameter remaining vertical during discharge, and the jet being uncontracted, let θ be the angle of the slit, z the height of the free

surface above the centre at the time t. In the element of time dt, the quantity discharged will be equal to

$$(2g)^{\frac{1}{2}} r\theta\, dt \int_{-r}^{s} (z-x)\ dx, \text{ or } \tfrac{2}{3}(2g)^{\frac{1}{2}} r\theta\, (z+r)^{\frac{3}{2}}\, dt,$$

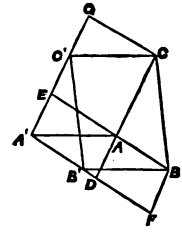
x and $x+dx$ being the distances from the centre of two horizontal planes intercepting an element of the slit. Equating this with the diminution of volume $\pi(r^2-z^2)\,dz$, there results for the time of emptying

$$t = \tfrac{2}{3}\pi \int_{-r}^{r} (r-z)\,(r+z)^{-\frac{1}{2}}\, dz \div (2g)^{\frac{1}{2}} r\theta = 4\pi\,(2r)^{\frac{1}{2}} \div (2g)^{\frac{1}{2}}\theta = 2\pi t' : \theta,$$

if t' is the time of falling freely down the height $2r$.

6196. (By the Editor.)—If ABC, A′B′C′ be two triangles with their corresponding sides equal and parallel; prove that (1) of the parallelograms formed by joining corresponding vertices one is equal to the other two; and hence (2) prove Euc. I. 47.

Solution by R. Rawson; C. Morgan, B.A.; *and others.*



The figures A′B, A′C, B′C are parallelograms;
also $A'C'CBB'A' = B'C + A'B'C' = A'B + A'C + ABC$,
and $\quad \triangle ABC = A'B'C'$;
therefore $\quad B'C = A'B + A'C$ .....(1).

It is readily seen that

$$B'C = BD + CE \text{ (Euc. I. 35)} \text{ .....(2)},$$

which seems to be necessary to prove Euc. I. 47 by the application of the neat property enunciated in (1). It is quite true that the form and position of the triangles ABC, A′B′C′ can be so constructed as to insure the parallelograms B′C, BD, CE to be the squares upon BC, AB, AC respectively. If this construction could be seen at a glance by an ordinary student of Euclid, then the proof of Euc. I. 47, by the above method, would be complete, and exceedingly simple.

Make the angle CAB a right angle, and the $\triangle ABC$ equal to the $\triangle BB'F$ in every respect, and draw AA′ equal and parallel to BB′ = BC.

Now, it is easily seen that $\triangle A'B'C' = \triangle ABC$; and that the parallelograms BD, CE are the squares upon AB, AC as required.

Because $\quad \angle B'BF = CBA$, and $\angle BB'F = ABB' = ACB$;
therefore $\quad \angle ABB' + CBA = CBA + ACB =$ a right angle $= \angle B'BC$;
therefore B′C is a square on BC; and (2) becomes $BC^2 = AB^2 + BC^2$.

The following is a simple application of the property in the question: Produce AC to D, making AD = AB; and AB to E, making AE = AC; through B draw FH equal and parallel to CD; upon HF draw the square A′H; and make $\quad FB' = A'C' = GC = AC$.
Then we have $A'B = GC' = CH = AB$, and the figure B′C′CB, being

equilateral and the rectangle is the square on BC; hence,

square AH = square BC + 4ΔBAC = square AB + square AC + 4ΔBAC;

therefore $BC^2 = AB^2 + AC^2$.

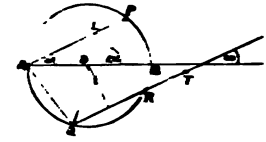
Hence, square AH = square AD + square AC + 2AH,

or $CD^2 = AD^2 + AC^2 + 2AC \cdot AD$. (Euc. II. 4.)

5600. By Christine Ladd.—Required the envelop of the Simson (or pedal) line.

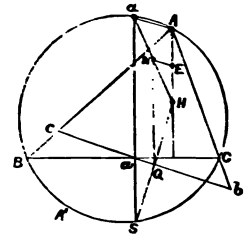
6290. By the Rev. F. D. Thomson, M.A.—Find the envelop of the axis of a parabola inscribed in a given triangle.

Solution by R. F. Davis, M.A., G. Eastwood, M.A.; *and others.*



1. The consideration of many theorems in which the three-cusped hypocycloid appears either as a locus or an envelop may be simplified by the following theorem:—"If A be a fixed point on a given circle, and PQ a chord having a fixed direction, the envelop of QR drawn parallel to AP is a three-cusped hypocycloid concentric with and circumscribing the circle." Let AOB be the diameter through A; α, α the inclinations to it of AP (or QR and PQ. Let OQ meet AP in L; then $\angle OQR = \angle ALQ = \angle AOQ$ $\omega = 2\alpha - 3\omega$. Thus the tangential polar equation to the envelop of QR is $p = a \sin 2\alpha - 3\omega$, which represents a three-cusped hypocycloid, concentric with and circumscribing the given circle.

A remarkable addition to this theorem is the following:—"If along QR points T, T' be taken such that QT = QT' = 2a, then T, T' are situate on the same curve." This may be shown either by analysis or geometry, and leads at once to the known property that the part of the tangent to a three-cusped hypocycloid intercepted by the curve is of constant length, and its middle point lies on the inscribed circle.



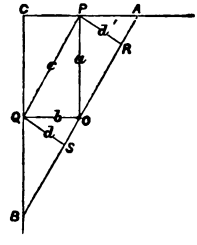
2. Let S be a point on the circle circumscribing the triangle ABC; S*a*, S*b*, S*c* perpendiculars upon the sides. Then *abc* is termed the Simson (or pedal) line corresponding to S (*Reprint*, Vol. XXIX., p. 80); and if *a* be the point where S*a* again meets the circle, and H the orthocentre, it is parallel to A*a*, and bisects SH in Q. Bisect HA, H*a* in E, N; then E, N, Q are points on the nine-points circle. Therefore, by the preceding theorem, since E is a fixed point, and QN (parallel to S*aa*) a fixed direction, the envelop of *abc* (parallel to EN) is a three-cusped hypocycloid concentric with and circumscribing the nine-points circle.

3. The Simson line being, *ipso facto*, the tangent at the vertex of the parabola inscribed in the triangle having S for its focus, then, if A′ be the extremity of the diameter through A, the axis of the parabola will be the straight line through S parallel to A′α. Therefore, A′ being a fixed point, and Sα a fixed direction, the envelop of the axis will be a three-cusped hypocycloid concentric with and circumscribing the circle about the triangle. [See p. 29 of this Volume of the *Reprint*.]

6226. (By Professor Matz, M.A.) — A post is a feet from the north wall, and b feet from the west wall, of a room whose sides meet each other at right angles; find the sides of the largest rectangular table that can be passed so as to touch at the same time the walls and the post, the surface of the table to remain horizontal all the time.

Solution by Prof. Genese, M.A.; R. Tucker, M.A.; *and others.*

Let the ground be the plane of (x, y), the axes of x and y being on the west and north walls respectively; and let (u, v) be the sides of a rectangle PQSR with the ends of u (P, Q) on x and y; then RS will envelop a curve obtained by eliminating α from



$$x\cos\alpha + y\sin\alpha = u\sin\alpha\cos\alpha + v,$$

$$-x\sin\alpha + y\cos\alpha = u\cos 2\alpha.$$

The condition that RS should pass round between the walls and the point O (b, a), is that O should *not* lie between the curve and the axes; or to get the greatest rectangle that O should lie on this curve;

$\therefore\ b\cos\alpha + a\sin\alpha = u\sin\alpha\cos\alpha + v,\quad -b\sin\alpha + a\cos\alpha = u\cos 2\alpha$ ... (1, 2).

The elimination of α gives the relation between u and v.

Again, uv is to be a maximum, therefore $\frac{du}{u} + \frac{dv}{v} = 0$. From (1), we have $du\sin\alpha\cos\alpha + dv + u\cos 2\alpha = -b\sin\alpha + a\cos\alpha$;

or, using (2), $du\sin\alpha\cos\alpha + dv = 0$, therefore $u\sin\alpha\cos\alpha - v = 0$..... (3).

From (1), (2), (3), we easily find $u = (a^2+b^2)^{\frac{1}{2}}$, $v = ab(a^2+b^2)^{-\frac{1}{2}}$.

Thus the length of the table is equal to the distance from the post to the intersection of the walls; and the area of the table is equal to the area enclosed, on the ground, by the walls and planes and parallels to them through the post.

[Let CA, CB be the walls of the room, and O the position of the post; then, drawing OP, OQ perpendicular to the walls, AB parallel to PQ, and PR, QS perpendicular to AB, the rectangle PQSR will be the required slab of table in its critical position. For, PQSR being the maximum rectangle, having its vertices P, Q on the sides CA, CB, that could be inscribed in the triangle ABC, no slab of greater area could attain to, much less pass through, the position in the figure. The position being supposed attained by the slab PQSR, then, on account of the perpendicularity of OP, OQ to

CA, CB, a small rotation in either direction of the slab round O as centre would disengage P and Q simultaneously from contact with the walls, and disengage, in consequence, the slab from its critical position.

Denoting by (c, d) the length and breadth of the slab, and by (a, b) the coordinates of the post, we have, evidently,

$$c = (a^2 + b^2)^{\frac{1}{2}}, \quad d = ab \,.\, c^{-1} = ab\,(a^2 + b^2)^{-\frac{1}{2}},$$

which give at once c and d in terms of a and b.]

6391. (By J. J. Walker, M.A.)—If O, A, B, C, D are any five points in space, prove that lines drawn from the middle points of BC, CA, AB respectively parallel to the connectors of D with the middle points of OA, OB, OC, meet in one point E, such that DE passes through, and is bisected by, the centroid of the tetrahedron OABC. [Quest. 6220 is a special case, in *two* dimensions, of the foregoing theorem in *three* dimensions.]

I. *Solution by* Professor Minchin, M.A.

Taking O as origin of vectors, and denoting the vectors to A, B, C, D by $\alpha, \beta, \gamma, \delta$ respectively, the equations of the three lines which are to be proved concurrent are

$$\rho = \tfrac{1}{2}(\alpha+\beta+\gamma) + \tfrac{1}{2}(x-1)\,\alpha - x\delta, \quad \rho = \tfrac{1}{2}(\alpha+\beta+\gamma) + \tfrac{1}{2}(y-1)\,\beta - y\delta,$$
$$\rho = \tfrac{1}{2}(\alpha+\beta+\gamma) + \tfrac{1}{2}(z-1)\,\gamma - z\delta.$$

But, if we make $x = y = z$, these are all satisfied by the vector $\frac{1}{2}(\alpha+\beta+\gamma) - \delta$. If this vector is ϖ, we have $\frac{1}{2}(\varpi+\delta) = \frac{1}{2}(\alpha+\beta+\gamma)$ = vector to centroid of tetrahedron OABC.

II. *Solution by* Professor Genese, M.A.

Let P, Q, R be the middles of BC, CA, AB; L, M, N the middles of the connectors; then the sides of the triangle LMN are equal and parallel to those of PQR. Therefore parallels through PQR to DL, DM, DN will meet in a point E such that EP = DL, EQ = DM, ER = DN; also EP must be parallel to DL; therefore DLEP is a parallelogram, and DE is bisected at the middle point of PL.

6173. (By Hugh McColl, B.A.)—If the statements a, b, f are true, then the statements c and x are true, or else the statements d, e, y are true. If the statement a is true, f false, and y true, then c is true and x false, or else d is false and e true. What inference may be drawn from these premises with respect to the truth or falsehood of the statements a, b, c, d, e, f, eliminating the statements x and y?

Solution by C. J. Monro, M.A.

Put A for $a\,(1-d'e)$; then, in Boole's notation (with accents),

$$fab\,(1-cx)\,(1-dey) = 0, \quad Af'\,(1-cx')\,y = 0.$$

The left-hand sides are positive class-terms, and may be combined by addition. Add them, making x and y successively 1 and 1, 1 and 0, 0 and 1, 0 and 0. The results are $fabc'(1-de)+Af'$, $fabc'$, $fab(1-de)+Af'c'$, fab. Take their product: this gives $fabc'(1-de)=0$ for the result of elimination. That is the whole effect of the data, irrespectively of x and y; but of course it may be expressed in various forms of inference. Perhaps the most natural is that, if f, a, and b are true, then c is true, or d and e, or all three. In fact, if we develope for fab in terms of c and de, we get $fab = \frac{2}{3}c + \frac{2}{3}c'de$.

6037. (By Prof. CROFTON, F.R.S.)—Prove that—

$$e^{x+D}\,F(x) = e^{1}e^{x}\,F(x+1), \quad \text{and} \quad e^{a(x+D)}\,Fx = e^{1a^2}e^{ax}\,F(x+a). \ \ldots$$

Solution by W. J. C. SHARP, M.A.; G. HEPPEL, M.A.; *and others.*

If (as in BOOLE's *Diff. Eq.*, 1st Ed., p. 395) $x+D$ be denoted by π, and D be resolved into D_1 and D_2, operating on x only as entering into $F(x)$ and π, and if $x+D_1 \equiv \pi_1$; then we have

$$\pi^n = \pi_1^n + \frac{n(n-1)}{1}\cdot\frac{\pi_1^{n-2}}{2} + \frac{n(n-1)(n-2)(n-3)}{1.2}\cdot\frac{\pi_1^{n-4}}{4} + \ldots + \frac{n!}{r!(n-2r)!}\cdot\frac{\pi_1^{n-2r}}{2^r};$$

therefore

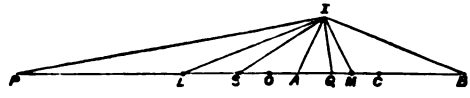
$$\begin{aligned} e^{a\pi} &= 1 + \frac{a\pi}{1} + \frac{a^2\pi^2}{2!} + \frac{a^3\pi^3}{3!} + \ldots + \frac{a^n\pi^n}{n!} + \&c. \\ &= 1 + \frac{a\pi_1}{1} + \frac{a^2\pi_1^2}{2!} + \frac{a^3\pi_1^3}{3!} + \ldots + \frac{a^n\pi_1^n}{n!} + \&c. \\ &\quad + \frac{a^2}{2} + \frac{a^3\pi_1}{1}\cdot\frac{1}{2} + \ldots + \frac{a^n\pi_1^{n-2}}{n-2!}\cdot\frac{1}{2} + \&c. \\ &\quad + \&c.\ \&c. \qquad + \frac{a^n\pi_1^{n-4}}{(n-4)!\,2!}\cdot\frac{1}{2^2} + \&c. \\ &= \left(1 + \frac{a^2}{2} + \frac{1}{2!}\cdot\frac{a^4}{2^2} + \&c.\right)\left(1 + \frac{a\pi_1}{1} + \frac{a^2\pi_1^2}{2!} + \&c.\right) = e^{1a^2}\cdot e^{a\pi_1}; \end{aligned}$$

therefore $e^{a\pi}\,F(x) = e^{1a^2}\cdot e^{a\pi_1}\cdot F(x) = e^{1a^2}e^{x}e^{aD_1}\,F(x) = e^{1a^2}e^{ax}\,F(x+a)$; and, if $a=1$, this becomes $e^{\pi}\,F(x) = e^{1}e^{x}\,F(x+1)$.

[For Prof. CROFTON's solution, see a paper in the *Quarterly Journal of Mathematics* for October, 1879.]

6386. (By W. S. McCAY, M.A.)—If two circles cut orthogonally, prove that (1) an indefinite number of pairs of points can be found on their common diameter such that either point has the same polar to one of the circles as the other point has to the other circle; and (2) the distance between such a pair of points subtends a right angle at an intersection of the circles.

Solution by Prof. GENESE, M.A.; H. MURPHY; *and others.*

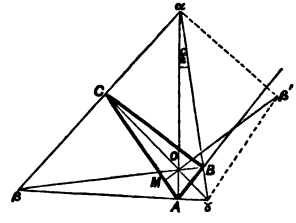


Let the circles AIB, LIM, with centres C and O, cut at right angles; and let S be any point in COL, and P and Q its conjugate points with respect to LIM and AIB. Then $\angle SIP = 2SIL$, $\angle SIQ = 2SIA$. Therefore $\angle PIQ = 2LIA = LIM =$ a right angle. Conversely, if PIQ be a right angle, the polar of P with respect to one circle, and the polar of Q with respect to the other, will coincide.

6331. (By J. W. RUSSELL, M.A.)—From the centres α, β, γ of the circles escribed to the triangle ABC, are drawn perpendiculars to the sides produced, so as to form a hexagon $\alpha\gamma'\beta\alpha'\gamma\beta$ whose opposite sides are parallel; show that the perpendiculars from α', β', γ' on the corresponding sides of the triangle meet in a point.

Solution by F. D. THOMSON, M.A.; J. O'REGAN; *and others.*

Here αA, βB, γC are the three perpendiculars of the triangle $\alpha\beta\gamma$. Let them meet in O. Draw $\alpha\beta'$ perpendicular to AB, $\gamma\beta'$ perpendicular to BC, &c.



Then, by the geometry,

$$\angle\gamma\alpha\beta' = \alpha\gamma\beta' = \tfrac{1}{2}B,$$

also $$\alpha O\gamma = \tfrac{1}{2}\pi + \tfrac{1}{2}B,$$

and hence β' is the centre of the circle round $\alpha O\gamma$. Hence

$$\angle\beta' O\gamma = \beta'\gamma O = \tfrac{1}{2}(\pi - C).$$

But if $\beta' O$ meet CA in M, $\angle\beta' O\gamma = COM$.

Therefore $\beta' OM$ is perpendicular to CA. Similarly, $\alpha' O$ is perpendicular to BC, and $\gamma' O$ is perpendicular to AB. Therefore the three perpendiculars from $\alpha'\beta'\gamma'$ on the corresponding sides of ABC meet in a point.

6319. (By W. S. B. WOOLHOUSE, F.R.A.S.) — On a given circle, of radius unity, a corner of a triangle is taken at a given distance (c) from the centre; the other two corners are taken at random on the surface; prove that the average area of all such triangles, in parts of the area of the circle, is

$$\frac{1}{\pi^2}\left(\tfrac{4}{5} + \tfrac{2}{3}c^2 - \tfrac{1}{6}c^4 + \tfrac{1}{36}c^6\right).$$

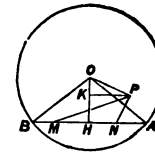
Solution by Professor SEITZ, M.A.

Let MNP be the triangle, P being fixed, AB the chord through M, N,

and O the centre of the circle. Draw OH perpendicular to AB, and PK to OH. Let

$OP = c$, $OA = 1$, $AM = x$, $MN = y$, $\angle AOH = \theta$,

$\angle POH = \phi$, $\theta' = \cos^{-1}c$, and $\cos\theta = c\cos\psi$.



Then we have area $MNP = \frac{1}{2}(\cos\theta - c\cos\phi)\,y$, when O and P lie on the same side of AB, and $= \frac{1}{2}(c\cos\phi - \cos\theta)\,y$, when they lie on opposite sides. An element of surface at M is $\sin\theta\,d\theta\,dx$, and at N it is $y\,d\phi\,dy$. When $\theta > \theta'$, O and P will lie on the same side of AB for all values of ϕ from 0 to π; but when $\theta < \theta'$, O and P will lie on opposite sides of AB from $\phi = 0$ to $\phi = \psi$, and on the same side from $\phi = \psi$ to $\phi = \pi$. The limits of x are 0 and $2\sin\theta$; and of y, 0 and x, and doubled. The results of the integrations with respect to ϕ must be doubled.

Hence, since the whole number of triangles is π^2, and the area of the circle is π, the average area is

$$\begin{aligned}
&\frac{4}{\pi^3}\int_0^{\theta'}\int_0^{\pi}\int_0^{2\sin\theta}\int_0^{x} \tfrac{1}{2}(\cos\theta - c\cos\phi)\,y\sin\theta\,d\theta\,d\phi\,dx\,y\,dy \\
&+\frac{4}{\pi^3}\int_{\theta'}^{\frac{1}{2}\pi}\int_0^{\psi}\int_0^{2\sin\theta}\int_0^{x} \tfrac{1}{2}(c\cos\phi - \cos\theta)\,y\sin\theta\,d\theta\,d\phi\,dx\,y\,dy \\
&+\frac{4}{\pi^3}\int_{\theta'}^{\frac{1}{2}\pi}\int_{\psi}^{\pi}\int_0^{2\sin\theta}\int_0^{x} \tfrac{1}{2}(\cos\theta - c\cos\phi)\,y\sin\theta\,d\theta\,d\phi\,dx\,y\,dy \\
&= \frac{8}{3\pi^3}\int_0^{\theta'}\int_0^{\pi}(\cos\theta - c\cos\phi)\sin^5\theta\,d\theta\,d\phi \\
&+\frac{8}{3\pi^3}\int_{\theta'}^{\frac{1}{2}\pi}\left\{\int_0^{\psi}(c\cos\phi - \cos\theta)\,d\phi + \int_{\psi}^{\pi}(\cos\theta - c\cos\phi)\,d\phi\right\}\sin^5\theta\,d\theta \\
&= \frac{8}{3\pi^2}\int_0^{\frac{1}{2}\pi}\sin^5\theta\cos\theta\,d\theta + \frac{16c^2}{3\pi^3}\int_0^{\frac{1}{2}\pi}(\sin\psi - \psi\cos\psi)(1 - c^2\cos^2\psi)^2\sin\psi\,d\psi \\
&= \frac{1}{\pi^2}\left(\tfrac{4}{5} + \tfrac{2}{3}c^2 - \tfrac{1}{6}c^4 + \tfrac{1}{28}c^6\right).
\end{aligned}$$

If all the corners of the triangle are taken at random on the surface of the circle, the average area of all the triangles is

$$\int_0^1 \frac{1}{\pi^2}\left(\tfrac{4}{5} + \tfrac{2}{3}c^2 - \tfrac{1}{6}c^4 + \tfrac{1}{28}c^6\right)c\,dc \div \int_0^1 c\,dc = \frac{35}{48\pi^2}.$$

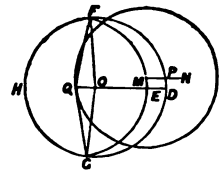
6298. (By A. Martin, M.A.)—If three equal circles be piled up at random on a horizontal plane, prove that the probability that the pile will stand, is

$$= \frac{1}{16} + \frac{3\sin^{-1}\frac{1}{4}}{8\pi} - \frac{9\sqrt{15}}{256\pi}.$$

Solution by Professor Seitz, M.A.

Let DFHG be the first circle, O its centre, M and N the centres of the

second and third circles. The pile will stand, if M and the middle point P of MN are in the circle O, and N is in the circle M.



If the circle M be moved parallel to itself so that MN will be constantly bisected by the circumference of the circle O, M will describe the arc GEF, whose centre is Q, and the area of GEFH will represent the number of favourable positions.

Let OD = 1, MN = x, $\angle$FOQ = θ, and let ϕ = the angle which MN makes with some fixed line. Then OQ = $\frac{1}{2}x = 2\cos\theta$, $x = 4\cos\theta$, $dx = -4\sin\theta\, d\theta$, and area GEFH = $2(\theta - \sin\theta\cos\theta)$.

The limits of x are 0 and 1; of θ, $\cos^{-1}\frac{1}{4} = \theta'$ and $\frac{1}{2}\pi$; and of ϕ, 0 and 2π. Hence, since the whole number of positions of the three circles is $16\pi^2$, the required probability is

$$\frac{1}{16\pi^2}\int_0^1\int_0^{2\pi} 2(\theta - \sin\theta\cos\theta)\, x\, dx\, d\phi$$

$$= \frac{4}{\pi}\int_{\theta'}^{\frac{1}{2}\pi} (\theta - \sin\theta\cos\theta)\sin\theta\cos\theta\, d\theta = \text{the result stated.}$$

6213. (By A. Martin, M.A.)—A cylinder, of radius r, *rolls* down the surface of another one, radius R, resting on a horizontal plane, the surface of both cylinders and plane being rough enough to secure perfect rolling. Determine the circumstances of the motion, the point of separation, and the path of the axis of the upper cylinder.

Solution by J. J. Walker, M.A.

Suppose C, C′, points in which the axes are cut by a plane perpendicular to their common direction, to be the centres of gravity of the cylinders, and the normal and tangential actions, N, T, of the lower on the upper, as also those, N′, T′, of the horizontal plane, on which the lower rolls, on it, to lie in that same vertical plane. Let θ be the angle which the line CC′ makes with the horizontal; ϕ, ϕ' the angle through which the cylinders have revolved, in the time t. Take the origin in the line of intersection of the vertical and horizontal planes above defined; and let xy be the co-ordinates of C, x'R those of C′. Then we have

$$x - x' = (r + R)\cos\theta,\ y = R + (r + R)\sin\theta,\ \dot{x}' = -R\dot{\phi}',\ r\dot{\phi} = R\dot{\phi}' \quad \ldots\ldots(1).$$

The movement of the upper cylinder is defined by

$$m\ddot{x} = N\cos\theta - T\sin\theta,\ m\ddot{y} = -mg + N\sin\theta + T\cos\theta,\ mk^2\ddot{\phi} = rT \quad \ldots\ldots(2),$$

that of the lower by

$$m'\ddot{x}' = -N\cos\theta + T\sin\theta + T',\ m'k'^2\ddot{\phi}' = R(-T + T') \quad \ldots\ldots(3).$$

From these three systems of equations, we get ($u = \cos\theta$)

$$\left\{(m + m')r^2R^2 + R^2mk^2 + r^2m'k'^2\right\}\ddot{x} = (r + R)\left\{R^2mk^2 + r^2m'(k'^2 + r^2)\right\}\ddot{u},$$

whence

$$\{\ \}x = \{\ \}\cos\theta + ct + c';\ \text{and}\ (r + R)\left\{mr^2[k^2 - r^2(1 - u^2)^{\frac{1}{2}}]\ddot{u} - A\ddot{\theta}\right\} = Ag,$$

where A stands for the coefficient of $\ddot{x}$ above. This latter equation, involving both $\ddot{\theta}$ and $(\dot{\theta})^2$, is insoluble actually; but, supposing it to give

$\theta = F(t)$, x and y would be determined as functions of the parameter t, and the locus of C be therefore found implicitly. Finally, substituting for $\ddot{x}$, $\ddot{y}$, θ, their values, in terms of t, in $\ddot{x}\cos\theta + (\ddot{y}+g)\sin\theta = 0$, the instant of separation of the cylinders would be determined.

6404. (By Sir James Cockle, F.R.S.)—Transform

$$y'' + 2(A\cot x + E\tan x)\,y' + \left\{Æ^2 - (A-E)^2\right\} y = 0 \quad \ldots\ldots\ldots\ldots (1)$$

into an equation of the form called by Boole binomial.

I. *Solution by* J. Hammond, M.A.; W. B. Grove, M.A.; *and others.*

The equation $y'' + 2(A\cot x + E\tan x)\,y' + Cy = 0$

is transformed into a binomial equation by taking the independent variable to be either $\cos x$ or $\sec x$, or any power of $\cos x$ or $\sec x$.

1. Writing $\cos x = X$, we have

$$y' = -\sin x \frac{dy}{dX}, \quad y'' = -X\frac{dy}{dX} + (1-X^2)\frac{d^2y}{dX^2},$$

and the given equation becomes

$$(1-X^2)\frac{d^2y}{dX^2} - X\frac{dy}{dX} - 2\left(AX + \frac{E(1-X^2)}{X}\right)\frac{dy}{dX} + Cy = 0;$$

which, by assuming $X = e^\theta$, is reduced to the form

$$y - \frac{(D+A-E-2)^2 - Æ^2}{D(D-1-2E)}\,e^{2\theta}y = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots (1).$$

2. Writing $\sec x = X$, we have

$$y' = X\frac{dy}{dX}\tan x, \quad y'' = X^3\frac{dy}{dX} + (X^2-1)\left(X\frac{d}{dX}\right)^2 y,$$

or, if $X = e^\theta$, $y' = \tan x\,Dy$, $y'' = (e^{2\theta}-1)\,D^2y + e^{2\theta}\,Dy$; and the equation becomes $(e^{2\theta}-1)\,D^2y + e^{2\theta}\,Dy + 2\left\{(A+E)(e^{2\theta}-1)\right\}Dy$

$$+ \left\{Æ^2 - (A-E)^2\right\} y = 0, \quad \text{or} \quad y - \frac{(D-2)(D-1+2E)}{(D-A+E)^2 - Æ^2}\,e^{2\theta}y = 0 \ldots (2).$$

3. The forms (1) and (2) are convertible, like those on p. 428 of Boole's *Differential Equations*, 2nd edition, 1865, by changing θ into $-\theta$; and the change of θ into $n\theta$ in either of them, would give the forms assumed by the equation when the independent variable is any power of $\cos x$ or of $\sec x$.

II. *Solution by the* Proposer.

Change the independent variable from x to arc tan x, then (1) is changed into

$$y'' + 2\frac{Ax^{-1} + (E+1)x}{1+x^2}\,y' + \frac{Æ^2 - (A-E)^2}{(1+x^2)^2}\,y = 0 \ldots\ldots (2).$$

Let $y = (1+x^2)^{-m} u$, then (2) becomes

$$u'' + 2\{Ax^{-1} + (E+1-2m)\,x\}\frac{u'}{1+x^2}$$

$$+\{Æ^2-(A-E)^2-2m(1+2A)+2m(2m-1-2E)\,x^2\}\frac{u}{(1+x^2)^2} = 0,$$

which will be binomial if

$$Æ^2-(A-E)^2-2m(2A+1) = 2m(2m-2E-1),$$

or $$(2m)^2+2(A-E)(2m) = Æ^2-(A-E)^2,$$

or $$Æ^2 = (A-E-2m)^2, \quad \text{or} \quad 2m = -A+E\pm Æ.$$

Substituting for m, we get

$$(1+x^2)u'' + 2\{Ax^{-1} + (A+1\mp Æ)\,x\}u'$$

$$+\left[\left\{Æ\mp(A+\tfrac{1}{2})\right\}^2-(E+\tfrac{1}{2})^2\right]u = 0,$$

which is a binomial.

6339. (By Prof. SYLVESTER, F.R.S.)—Understanding by an Algebraical Perimeter of a rectilinear figure, the sum of the sides, each taken with the positive or negative sign; prove that the necessary and sufficient condition of all the curves described by points rigidly attached to the connecting link in a 3-bar link-work being unicursal is that one of the algebraical perimeters of the quadrilateral (real or impossible) formed by the three links and the line of centres shall be equal to zero.

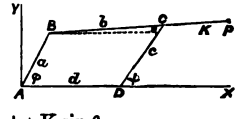
[Professor SYLVESTER states that ROBERTS' two cases of inverse conical 3-bar motion are special cases of unicursal 3-bar motion.]

Solution by SAMUEL ROBERTS, M.A.

Professor CAYLEY has shown (*Mathematical Society's Proceedings*, Vol. IV., pp. 109, 110) that the condition specified in the question is sufficient, and notices the cases of reduction of order.

To show the necessity of the condition, I consider that in general the curves described by points rigidly connected with the traversing bar, have a one-to-one correspondence, and have therefore the same deficiency. We may therefore regard a point on the traversing bar as describing the locus, and since the extremities of the links which rotate describe exceptional loci, we will suppose the describing point different from these.

The locus described is evidently symmetrical about the line of centres. Moreover, since the circular points are triple points, and there are three double points on the line of centres, we must, in order to have a unicursal curve, have on this line a contact of two branches whose common tangent is perpendicular to the line of centres. For we cannot have four double points on the line of centres identically, nor one double point finitely distant from that line; since, then, there would be another double point on the other side of it. Now, taking A as origin, and the line of centres as the axis of x (see figure), we have



$$x = a\cos\phi + (b+K)\cos\theta,$$

$$y = a\sin\phi + (b+K)\sin\theta,$$

$$x = c\cos\psi + K\cos\theta + d, \quad y = c\sin\psi + K\sin\theta;$$

and the equation of the locus is obtained from these in the form

$$y^2\{K[x^2+y^2+(b+K)^2-a^2]-(b+K)[(x-d)^2+y^2+K^2-c^2]\}^2$$
$$+\{K(x-d)[x^2+y^2+(b+K)^2-a^2]-(b+K)x[(x-d)^2+y^2+K^2-c^2]\}^2$$
$$=4d^2K^2(b+K)^2y^2.$$

The three double points on the axis of x are determined by

$$K(x-d)\{x^2+(b+K)^2-a^2\}-(b+K)x\{(x-d)^2+K^2-c^2\}=0\ldots(a);$$

and since the coefficient of y^2 must vanish, for $x=$ root of (a), we have also

$$K\{x^2+(b+K)^2-a^2\}-(b+K)\{(x-d)^2+K^2-c^2\}=0\ldots\ldots\ldots(b),$$
$$\pm 2dK(b+K).$$

Multiplying (b) by x, and subtracting (a) from the result, we have

$$Kd\{x^2+(b+K)^2-a^2\}\pm 2dK(b+K)x=0,$$

or

$$Kd\{[x\pm(b+K)]^2-a^2\}=0, \text{ or } x=\mp(b+K)\pm a.$$

Substituting this value of x in (b), and remarking that $-(b+K)$ corresponds to $+2dK(b+K)x$, and $+(b+K)$ to $-2dK(b+K)x$, we get the conditions $(b+d\mp a)^2-c^2=0$ or $(b-d\pm a)-c^2=0$. We disregard the factors K, $b+K$, d as irrelevant and corresponding to the excluded extremities of the rotating links or to an evanescent bar.

6384. (By Dr. HOPKINSON, F.R.S.)—A heavy wire of uniform section is carried on a series of supports in the same horizontal plane, L_r being the bending moment at the r^{th} point of support, l_r the distance between the $(r-1)^{th}$ and r^{th} support, and m the mass of the wire per unit length; prove that $L_{r-1}l_r+2L_r(l_r+l_{r-1})+L_{r+1}l_{r+1}=\frac{1}{4}mg(l_r^3+l_{r+1}^3)$.

Solution by Professor MINCHIN, M.A.

This is the famous "Equation of Three Moments," restricted to the case of a uniformly loaded continuous beam. Demonstrations of the equation will be found in various works on Applied Mechanics; among others, see those of BRESSE and COLLIGNON.

Briefly the proof is this,—Let A, B, C be any three consecutive points of support; let M_1, M_2, M_3 be the bending moments at these points; A_0 and A_1 the shearing forces just behind and just in front of A; B_0 and B_1, C_0 and C_1 the shearing forces just behind and just in front of B and C respectively; $l=AB$, $l'=BC$; w and w' the loads per unit length in the spans AB and BC.

Now the bending moment at any section is equal to $\frac{EI}{\rho}$, in the well-known notation of the subject. First take B as origin, BA as axis of x, and the downward vertical as axis of y. Then, if x, y are the coordinates of any point of the beam between B and A,

$$EI\frac{d^2y}{dx^2}=M_2-B_0x+\tfrac{1}{2}wx^2,$$

[illegible]

[illegible]

[illegible]

6375. (By Prof. CROFTON, F.R.S.)—If a heavy particle be moving, up or down, on a curve (rough or smooth), show that, if it leaves the curve, the parabola described has contact of the second order with the curve.

Solution by C. J. MONRO, M.A.; G. F. WALKER, M.A.; *and others.*

The particle will quit the curve on the concave (or convex) side as soon as the curvature of the orbit which it would describe if the curve had just come to an end has become greater (or less) than the curvature of the curve, and friction will only promote or prevent the satisfying of the condition. In either case when the particle quits the curve the curvature of its free orbit will be equal to that of the curve; and this is another way of stating the proposed theorem. The free orbit need not be a parabola, for the above holds good whatever the forces. *E.g.*, if there are none, the particle will quit the curve at a point of inflexion or a cusp only.

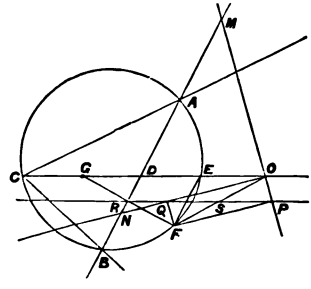
6245. (By Prof. COCHEZ.)—Démontrer que l'enveloppe des axes des coniques tangentes à deux droites en deux points donnés est une parabole.

I. *Solution by* R. F. DAVIS, M.A.; Prof. GENESE, M.A.; *and others.*

Let OP, OP′ be two tangents to a conic; PG, P′G′ the corresponding normals measured to the axis; and CD, CD′ the corresponding semi-conjugate diameters. Then OP : OP′ = CD : CD′, and PG : CD :: b : a :: P′G′ : CD′; wherefore OP : OP′ :: PG : P′G′. Now P, P′ are fixed points, and PG, P′G′ fixed directions; the envelop therefore of the axis GG′ is a parabola touching the two normals and the chord of contact (*Reprint*, XXII., p. 85). It may be noticed that the orthocentres of the triangles formed by any three of the five straight lines—the chord of contact, the two normals, and the two axes—all lie upon the directrix CO.

II. *Solution by* Prof. TOWNSEND, F.R.S.; Prof. MATZ, M.A; *and others.*

If, in the figure, AC and BC be the two right lines, touched at A and B by all the conics of the system, D the middle point of the common chord of contact AB, E the second point at which the line of centres CD of the system is met by the circle ABC, and EF the chord of ABC through E parallel to AB; the parabola of which F is the focus and CD the directrix is that of the question.



For, if O, on the line of centres CD, be the centre of any conic of the system, M and N the two points at which any pair of its conjugate diameters OM and ON intersect the chord of contact AB, and G the reflexion of F with respect to the axis AB, which point lies manifestly on

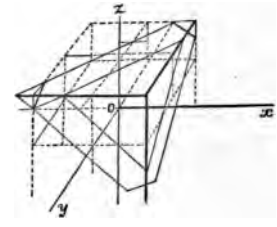
the line of centres CD at the distance from D equal and opposite to that of E; then, since, by a familiar property of the circle, DE, and therefore DG, is the third proportional to CD and AD or BD, and since consequently, by a fundamental property of central conics, the rectangle DM . DN is equal to the rectangle DG . DO, the four points G, M, N, O lie always on a circle, which, when OM and ON are at right angles to each other, and therefore the axes of the conic, has its centre on AB, and passes consequently through F as well as through G; hence, the feet P and Q of the perpendiculars FP and FQ from F upon the axes OM and ON of the conic are collinear at once with the middle points R and S of the two rectilinear segments FG and FO, and lie consequently on the fixed right line RS parallel to CD, and lying midway between it and F; and as consequently, of the two variable right angles FPM and FQN, two of the sides FP and FQ turn round the fixed point F, while the two vertices P and Q move on the fixed right line RS, therefore the two remaining sides PM and QN envelop the parabola of which F is the focus and RS the tangent at the vertex, that is, the parabola of which F is the focus and CD the directrix; and therefore, &c.

6382. (By ELIZABETH BLACKWOOD.)—If x, y, z be each taken at random between a and $-a$, what is the chance that $x+y+z$ will be between b and $-b$? Explain any easy experimental method by which your result may be verified. [Miss BLACKWOOD states that, taking a and b each $=1$, she has found the event to happen 65 times in 100 trials.]

Solution by G. F. WALKER, M.A.; D. EASTWOOD, M.A.; *and others.*

Consider the section of the sides of the cube $x = \pm a$, $y = \pm a$, $z = \pm a$ by the two planes $x+y+z = \pm b$.

(1) Let $b > a$. The planes will cut off from the cube two tetrahedrons whose edges are $3a-b$, and the combined volume of these is $2 \cdot \frac{1}{6}(3a-b)^2$, and taking xyz as the coordinates of a point, and all points in the cube being equally likely, the chance is the ratio of the volume of the cube between the planes and the whole cube



$$= \frac{(2a)^3 - 2 \cdot \frac{1}{6}(3a-b)^3}{(2a)^3}$$

$$= \frac{3(2a)^3 - 2(3a-b)^3}{24a^3} = \frac{b^3 - 9ab^2 + 27a^2b - 3a^3}{24a^3}.$$

(2) Let $b < a$. The planes now cut the cube in hexagons, and the volume now cut off by one of them is

$$\tfrac{1}{6}[(3a-b)^3 - 3(a-b)^3],$$

and chance

$$= \frac{(2a)^3 - \frac{1}{3}[(3a-b)^3 - 3(a-b)^3]}{(2a)^3}$$

$$= \frac{24a^3 - [24a^3 - 18a^2b + 2b^3]}{3(2a)^3} = \frac{18a^2b - 2b^3}{24a^3}.$$

If $a = b$, both of these become $\frac{2}{3}$, which is very near to $\frac{65}{100}$.

6353. (By the Rev. J. VENN, M.A.)—All x is either y and not z, or z and not y. All wx is either both y and z or neither of them. All xy which is not w is z. No x that is neither y nor w is z. Show that under these conditions there is no x.

Solution by HUGH McCOLL, B.A.

Speaking throughout of some *one* originally unclassed individual, let x, y, z respectively denote the statements,—it belongs to the class x, it belongs to the class y, it belongs to the class z. Then the data will be expressed by the complex statement

$$(x : yz' + zy')\,(wx : yz + y'z')\,(xyw' : z)\,(xy'w' : z).$$

Let $f(x)$ denote this statement. We have to show that $xf(x) : 0$, that is, that $f(x) : x'$. Now $xf(x) : f(1)$, and since every implication $\alpha : \beta$ has a disjunction consequent $\alpha' + \beta$, we get

$$f(1) : (yz' + zy')\,(w' + \overline{yz + y'z'})\,(y' + \overline{w} + z)\,(y + \overline{w} + z').$$

We over-line $yz + y'z'$ as zero, because it is the denial of the first factor $yz' + zy'$; we then over-line w as the denial of the (then) second factor w'; the product of the last two factors is $yz + y'z'$, which is the denial of $yz' + zw'$. We thus find (without the trouble of multiplication) that $xf(x) : f(1) : 0$, that is, that $f(x) : x'$.

6357. (By J. E. A. STEGGALL, M.A.)—Find in how many ways a given number n may be split up into the sum of 1, 2, 3 ... different numbers. *Example.*—In how many ways can 34 be split up into 4 different numbers?

Solution by W. A. WHITWORTH, M.A.; Prof. GENESE, M.A.; *and others.*

Let P_r^n denote the number of r-partitions of n; then we have to cast out from P_4^{34} all the partitions in which the four numbers are not all "different." But those in which there are two *ones*, two *twos*, two *threes*, are P_2^{32}, P_2^{30}, P_2^{28}, and so on. Hence the total number of doublets among the partitions is $P_2^{32} + P_2^{30} + P_2^{28} + \ldots + P_2^2$.

But, if we were to reject all these, we should be rejecting *twice* every partition which consisted of two doublets. The number of such partitions is evidently P_2^{17}. Replacing these, we get the final result

$$P_4^{34} - \{P_2^{32} + P_2^{30} + \ldots + P_2^2\} + P_2^{17}.$$

Now (*Choice and Chance*, p. 94), $P_4^{34} = \frac{34^3 + 3 \cdot 34^2 - 4}{144} = 297$; and P_2^{2r} or $P_2^{2r+1} = r$; hence the required number is

$$297 - (16 + 15 + 14 + \ldots + 1) + 8 = 169.$$

6358. (By R. Tucker, M.A.)—If O, P are the equidistant angle-point and the orthocentre of a triangle, and ρ_1, ρ_2, ρ_3 the radii of circles about OPA, OPB, OPC respectively; show (1) that

$$OP^2(\rho_1^{-2}+\rho_2^{-2}+\rho_3^{-2}) = 8[1-\cos(A-B)\cos(B-C)\cos(C-A)];$$

and (2) that, if any angle, as C, = 60°, then four of the points, as A, B, O, P, are concyclic. [The terms "equidistant angle-point," and "equidistant side-point," Mr. Tucker proposes to use in place of "centre of circumscribed circle" and "centre of inscribed circle" of a triangle.]

Solution by G. Heppel, M.A.; A. McMurchy, B.A.; *and others.*

Here we have $\angle OAP = B-C$, therefore $2\rho_1 \sin(B-C) = OP$, whence

$$OP^2(\rho_1^{-2}+\rho_2^{-2}+\rho_3^{-2}) = 4[\sin^2(B-C)+\sin^2(C-A)+\sin^2(A-B)]$$
$$= 8[1-\cos(A-B)\cos(B-C)\cos(C-A)].$$

If $C=60°$, $A+B=2c$, therefore $B-C=C-A$, therefore $\angle OAP = \angle OBP$, and A, B, O, P are concyclic.

6340. (By Prof. Townsend, F.R.S.)—The normal at a variable point on a fixed surface being supposed to intersect in every position a fixed right line in the space of the surface; prove, from the requisite equation of condition, that—

(*a*) Whatever be the position of the line in the space, the curve locus of the variable point is the intersection with the surface of another of the same order, passing through its several points of intersection with the line, and through the several points at which its tangent planes are perpendicular to the line.

(*b*) When the line at any of the aforesaid points of intersection is itself a normal to the surface, its point of normal intersection is a crunodal double point of the aforesaid curve, the two tangents at which intersect at right angles, and coincide in direction with the two principal tangents to the surface at the point.

I. *Solution by* C. J. Monro, M.A.; G. Eastwood, M.A.; *and others.*

Let $U = 0$ determine the surface, and take the right line for z-axis. One of the equations to the normal gives the condition

$$\frac{dU}{dy}x - \frac{dU}{dx}y = 0 \quad \ldots\ldots(1);$$

whence (*a*) follows immediately.

(*b*) Measure x and y along principal tangents; then U is of the form $cz+ax^2+by^2+T$, and (1) becomes $xy+T'=0$, where T and T′ are of the same order of magnitude as x^3, y^3, z^3, at most. These give a curve touching the principal tangents where they cross.

II. *Solution by the* PROPOSER; J. L. MCKENZIE, B.A.; *and others.*

To prove (*a*).—If $\phi = 0$ be the equation of the surface, referred to any rectangular triad of coordinate planes intersecting at any origin on the fixed line in the space, x, y, z the coordinates of the variable points in any position on the surface, and l, m, n those of any fixed points on the line; then since, by hypothesis, the line through the origin, the cosines of whose direction angles are proportional to $\frac{d\phi}{dx}, \frac{d\phi}{dy}, \frac{d\phi}{dz}$, must, in every position of xyz on the surface, be complanar with the two the cosines of whose direction angles are proportional to x, y, z and to l, m, n, and since consequently the determinant of the nine cosines must vanish, we have, consequently, between the coordinates x, y, z, the equation

$$(mz-ny)\frac{d\phi}{dx} + (nx-lz)\frac{d\phi}{dy} + (ly-mx)\frac{d\phi}{dz} = 0;$$

which being of the same order with ϕ, and being satisfied by the two systems of values of x, y, z for which, respectively,

$$x : y : z = l : m : n \quad \text{and} \quad \frac{d\phi}{dx} : \frac{d\phi}{dy} : \frac{d\phi}{dz} = l : m : n;$$

therefore &c., as regards (*a*).

To prove (*b*).—Taking, in that case, the line itself as axis of z, and the tangent plane at its point of normal intersection with the surface as plane of xy, so that, consequently,

$$\phi = mz + \tfrac{1}{2}(ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy) + \&c.;$$

then, since in the above equation $l = 0$ and $m = 0$, the equation becomes reduced to

$$y\frac{d\phi}{dx} - x\frac{d\phi}{dy} = 0;$$

or, from the value of ϕ in this case, to

$$h(y^2-x^2) + (a-b)xy + \&c. = 0,$$

which shows that the origin is a connodal double point on the curve in question, the tangents at which

$$h(y^2-x^2) + (a-b)xy = 0$$

are at right angles to each other, and bisect internally and externally the angle

$$ax^2 + 2hxy + by^2 = 0$$

determined by the two inflexional tangents to the surface at the point; and therefore &c., as regards (*b*).

N.B.—The known directions of the two "lines of curvature" through any point of a surface of any order are obviously given immediately by this latter property.

6396. (By C. LEUDESDORF, M.A.)—Show that the square of the tangent from any point P to the circle inscribed in the triangle ABC is

$$\frac{a \cdot AP^2 + b \cdot BP^2 + c \cdot CP^2}{a+b+c} - r(r+2R),$$

r being the radius of this circle, and R that of the circumscribed circle.

Solution by G. TURRIFF, M.A.; E. W. SYMONS, M.A.; *and others.*

If O be the centre of the inscribed circle, then (TOWNSEND's *Mod. Geom.*, Art. 102) $OP^2 = \frac{a \cdot AP^2 + b \cdot BP^2 + c \cdot CP^2}{a+b+c} - \frac{abc}{a+b+c}$,

$2Rr = 2 \cdot \frac{abc}{4\Delta} \cdot \frac{\Delta}{s} = \frac{abc}{a+b+c}$, (Tangent)$^2 = OP^2 - r^2$ = &c.

[It is remarked that Quest. 6392 may be proved in a similar way.]

5341. (By J. W. SHARPE, M.A.)—Prove that, if α, β, γ, ρ be any four vectors, then $V\rho\alpha \cdot V\beta\gamma + V\rho\beta \cdot V\gamma\alpha + V\rho\gamma \cdot V\alpha\beta$ is also a vector.

Solution by CHRISTINE LADD; *the* PROPOSER; *and others.*

$$\alpha V\beta\gamma + \beta V\gamma\alpha + \gamma V\alpha\beta = 3S\alpha\beta\gamma;$$

for the vector of the left side is zero, since it is

$$\gamma S\alpha\beta - \beta S\gamma\alpha + \alpha S\beta\gamma - \gamma S\alpha\beta + \beta S\gamma\alpha - \alpha S\beta\gamma;$$

and the scalars of the two sides are clearly the same. Operate upon both sides with $S\rho$, and we get $S(\rho\alpha V\beta\gamma + \rho\beta V\gamma\alpha + \rho\gamma V\alpha\beta) = 0$;

hence $S(V\rho\alpha \cdot V\beta\gamma + V\rho\beta \cdot V\gamma\alpha + V\rho\gamma \cdot V\alpha\beta) = 0$;

hence $V\beta\gamma \cdot V\alpha\rho + V\gamma\alpha - V\beta\rho + V\alpha\beta \cdot V\gamma\rho$ denotes a vector.

6320. (By ELIZABETH BLACKWOOD.)—Given that $ax + by$ is less than c and greater than d, and that $cx + dy$ is less than a and greater than b; required the limits of x and y, a, b, c, d being all positive.

Solution by Prof. TANNER, M.A.; J. A. KEALY, M.A.; *and others.*

We have $ax + by = (c-d)\,t + d$, $cx + dy = (a-b)\,t' + b$; where t, t' are positive proper fractions; hence

$$(ad-bc)\,x = (c-d)\,dt + d^2 - (a-b)\,bt' - b^2.$$

Take $ad-bc$ to be positive; then the greatest value of x is given by $t=1$, $t'=0$, the least by $t=0$, $t'=1$; hence

$$\frac{cd-b^2}{ad-bc} > x > \frac{d^2-ab}{ad-bc};$$ so, too, $$\frac{a^2-cd}{ad-bc} > y > \frac{ab-c^2}{ad-bc}.$$

If $ad-bc$ be negative, the x limits are transposed, and likewise those for y. If $ad-bc=0$, the data are contradictory, unless $a, b = c, d$, when they are insufficient.

5624. (By Professor SYLVESTER, F.R.S.)—If X_j is to be a rational homogeneous function of the j^{th} order of i independent quantities $a, b, c, \ldots k, l$, and is to satisfy the partial differential equation

$$(A\delta_a + B\delta_b + \ldots + L\delta_l)\, X_j = 0,$$

where A, B, ... L are homogeneous linear functions of $a, b, \ldots, l$, not absolutely independent, but connected by a single homogeneous linear equation with one another; prove that the number of arbitrary parameters in X_j will in general be the number of solutions in positive integers (zero counted in as positive) of the simultaneous equations

$$u_0 + u_1 + u_2 + \ldots + u_{i-1} = j\tfrac{1}{2}v_0 + u_1 + 2u_2 + \ldots (i-1)\, u_{i-1} = (i-1)\, \tfrac{1}{2}jv_0 + v_1 = 1.$$

Or, again, suppose $[(a-b)\,\delta_c + (b-c)\,\delta_a + (c-a)\,\delta_b]\, X_2 = 0$, the complete value of X_2, viz., $k\,(a^2+b^2+c^2) + k'(ab+ac+bc)$, contains 2 parameters k, k', which will be found to agree with the rule; and so, if in the above we write X_{2i} or X_{2i+1} in place of X_2, it will be found (still in accordance with the rule) that there will be $i+1$ arbitrary parameters in each of these forms.

Example:—Let $i=2$, then, if $(\lambda A\delta_a - \mu A\delta_b)\, X_j = i$, $X_j = k\,(\mu a + \lambda b)^j$, and contains just *one* arbitrary parameter k, which agrees with the rule.

Solution by the PROPOSER.

This is a simple consequence of the rule—*re* Invariants—given in the *Philosophical Magazine*, some time ago, but is far from being obviously so.

By linear substitutions, the equation may be changed *in general* into

$$(A'\delta_{B'} + B'\delta_{C'} + \ldots + K'\delta_{L'})\, X'_j = 0,$$

where X′ is now regarded as a function of A′, B′, ... L′; and then, by some simple, but a trifle subtle argumentation, it may be shown that the number of arbitrary parameters required is the total number of invariants and covariants of the order j to a quartic of the $(i-1)$ degree, which by virtue of a known theorem is the number of solutions of the equations in the question. I think it rather valuable as carrying the invariantive theory into a higher sphere.

It may be observed that $v_0 + v_1 = 1$ simply says that v_0 is to be 0 or 1 according as $(i-1)j$ is even or odd.

It would probably be difficult to prove the theorem without having recourse to the theory of invariants, as one would never have thought otherwise of the transformation into $A'\delta_{B_1} + \ldots$, and even then it is not easy (without the light of this theory) to see that the general number of solutions might not exceed the number stated.

6188. (By Professor SYLVESTER, F.R.S.)—If in any persymmetrical matrix of the n^{th} order, each column in succession calling its elements $c_1, c_2 \ldots c_n$, be replaced by $0, c_1, 2c_2 \ldots (n-1)\, c_n$, show that the sum of the n determinants thus formed is zero. For example, show that

$$\begin{vmatrix} 0 & b & c & d \\ a & c & d & e \\ 2b & d & e & f \\ 3c & e & f & g \end{vmatrix} + \begin{vmatrix} a & 0 & c & d \\ b & b & d & e \\ c & 2c & e & f \\ d & 3d & f & g \end{vmatrix} + \begin{vmatrix} a & b & 0 & d \\ b & c & c & e \\ c & d & 2d & f \\ d & e & 3e & g \end{vmatrix} + \begin{vmatrix} a & b & c & 0 \\ b & c & d & d \\ c & d & e & 2e \\ d & e & f & 3f \end{vmatrix} = 0.$$

Solution by H. STABENOW, M.A.; J. HAMMOND, M.A.; *and others.*

a, b, c, &c. may be considered as s_0, s_1, s_2, &c., the sums of the powers of α, β, γ, &c., the roots of an equation. The persymmetrical determinant of the n^{th} order will then be (SALMON'S *Higher Algebra*, 3rd ed., p. 22)

$$\Sigma\,(\alpha-\beta)^2\,(\beta-\gamma)^2\,(\alpha-\gamma)^2 \ldots \text{ to } \tfrac{1}{2}n\,(n-1) \text{ factors.}$$

Now, operating on both sides with $\left(\frac{d}{da}+\frac{d}{d\beta}+\ldots\right) \equiv \Delta$, and observing that $\Delta s_r = r\Delta s_{r-1}$, the result follows at once, since

$$\Delta\Sigma\,(\alpha-\beta,^2\,(\beta-\gamma)^2\,(\alpha-\gamma)^2 \ldots = 0.$$

6207. (By E. ANTHONY, M.A.) — In a plane triangle ABC, if $\frac{1}{2}\Sigma \sin^2 A \equiv \sigma$, and $\sigma - \sin^2 A = \sigma_1$, &c., prove that

$$\sigma_2\sigma_3 + \sigma_3\sigma_1 + \sigma_1\sigma_2 = \sin^2 A \sin^2 B \sin^2 C.$$

Solution by A. MCMURCHY, B.A.; K. KNOWLES, B.A., L.P.C.; *and others.*

Writing *sines* for sides in the well known zero-homogeneous expressions for cos A, we have $\sigma_1 = \sin B \sin C \cos A$;

therefore $\sigma_2\sigma_3 + \ldots + \ldots$

$= \sin A \sin B \sin C (\sin A \cos B \cos C + \sin B \cos C \cos A + \sin C \cos A \cos B)$

$= \sin^2 A \sin B \sin C (\cos A + \cos B \cos C)$

$= \sin^2 A \sin^2 B \sin^2 C.$

6216. (By R. TUCKER, M.A.) — O is the point of projection of two equi-horizontal-range particles, and (OP, OP′), (OQ, OQ′) the normal chords and diameters of curvature at O respectively; prove that, if the initial velocity be the same for both, PQ′ is parallel to QP′.

Solution by A. ANDERSON, M.A.; *the* PROPOSER; *and others.*

The equation to the path is

$$y = x \tan \alpha - \frac{gx^2}{2u^2 \cos^2 \alpha}, \text{ or } gr \cos^2\theta = 2u^2 \cos\alpha \sin(\alpha-\theta) \quad \ldots(1).$$

Since OP is a normal chord ($\theta = \alpha - \frac{1}{2}\pi$), therefore

$$\text{OP} = \frac{2u^2}{g} \cot\alpha \operatorname{cosec}\alpha = 4h \cot\alpha \operatorname{cosec}\alpha,$$

$$\text{OP}' = \frac{2u^2}{g} \tan\alpha \sec\alpha = 4h \tan\alpha \sec\alpha.$$

Now OQ is the diameter of curvature at O, hence we have

$$OQ = \frac{4SO^2}{SY} = \frac{4h^2}{h\cos\alpha} = 4h\sec\alpha \text{ (SY perpendicular on tangent at O).}$$

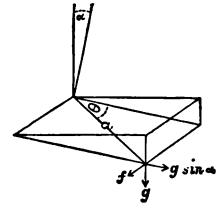
Similarly $OQ' = 4h \operatorname{cosec} \alpha$;

therefore $OP \cdot OP' = OQ \cdot OQ'$; hence PQ′ is parallel to QP′.

6259. (By G. S. Carr, B.A.)—A railway carriage door of width $2a$ and mass m, with the axis through its hinges inclined α to the vertical in a plane at right angles to the rails, hangs open at rest. The carriage suddenly moves forward with an initial velocity v, and a constant acceleration afterwards f. Show that—(1) if the door slams, the force of the blow on the jamb is $\frac{1}{3}m\left[v^2+\frac{8}{3}a(f-g\sin\alpha)\right]^{\frac{1}{2}}$; and (2) the time of a small oscillation of the door about the position of equilibrium, is

$$4\pi(\tfrac{1}{3}a)^{\frac{1}{2}}(f^2+g^2\sin^2\alpha)^{-\frac{1}{4}}.$$

Solution by the Proposer.



Let u be the initial velocity of the centre of inertia of the door; w, w' its initial and final angular velocities; P, P′ the corresponding impulsive forces upon the hinges and the jamb. Then $P = mu$, $Pa = mk^2w$, and $u+aw = v$;

therefore $w = va \div (a^2+k^2)$.....(1).

Reduce the axis of the door to rest by applying to the centre of inertia the reversed acceleration f and the reversed initial velocity v. The initial velocity of the centre of inertia will then become aw in the direction opposite to u. The equation of *vis viva* is now

$$m(a^2+k^2)(w'^2-w^2) = 2(mfa-mga\sin\alpha), \text{ and } k^2 = \tfrac{1}{3}a^2;$$

$$\therefore\ w' = \left\{w^2+\frac{3}{2a}(f-g\sin\alpha)\right\}^{\frac{1}{2}} = \frac{3}{4a}\left\{v^2+\frac{3}{8a}(f-g\sin\alpha)\right\}^{\frac{1}{2}}, \text{ by (1).}$$

By moments about the axis we have $2aP' = m(a^2+k^2)w'$;

therefore $P' = \frac{2}{3}maw' = \frac{1}{3}m\left\{v^2+\frac{8}{3}a(f-g\sin\alpha)\right\}^{\frac{1}{2}}$.

Secondly, to find the time of oscillation. The equation of motion is

$$(k^2+a^2)\ddot{\theta} = fa\cos\theta - g\sin\alpha\, a\sin\theta \text{.....(2).}$$

Put $\ddot{\theta} = 0$ to find the position of equilibrium; thus

$$f\cos\theta_0 - g\sin\alpha\sin\theta_0 = 0 \text{.....(3).}$$

For a small oscillation put $\theta = \theta_0+\phi$ in (2); then we have

$$\tfrac{4}{3}a\ddot{\phi} = f(\cos\theta_0-\phi\sin\theta_0) - g\sin\alpha(\sin\theta_0+\phi\cos\theta_0)$$
$$= -\phi(f\sin\theta_0+g\sin\alpha\cos\theta_0), \text{ by (3).}$$

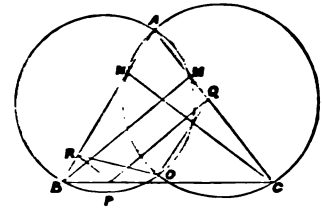
Eliminate θ_0 by (3); thus $\ddot{\phi}+\frac{3}{4}a\left\{f^2+g^2\sin^2\alpha\right\}^{\frac{1}{2}}\phi = 0$.

Therefore time of oscillation $= 4\pi(\tfrac{1}{3}a)^{\frac{1}{2}}\left\{f^2+g^2\sin^2\alpha\right\}^{-\frac{1}{4}}$.

6365. By H. Murphy.—**If from a variable point on any side of a triangle two lines are drawn making given angles with it; prove that the circle through the points of intersection with the other sides and opposite vertex passes also through a fourth fixed point.**

Solution by Prof. Genese, M.A.; G. Heppel, M.A.; *and others.*

Let PQ, PR be two of the lines in question, and let BM, CN be parallels thereto; also let the circles ABM, ACN intersect in O; then the triangles OCM, ONB are similar, and CQ : QM = CP : BP = NR : RB; hence the triangles OCQ and ONR are similar; therefore $\angle NOR = COQ$, and $\angle QOR = CON$ = supplement of A. Thus the circle AQR passes through O.



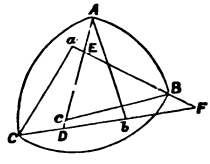
6214. (By G. Heppel, M.A.)—**If the sides of a plane triangle be circular arcs of equal radius, find a simple necessary relation between the sides and angles of the triangles.**

Solution by J. J. Walker, M.A.

Let a, b, c be the centres of the equiradial arcs BC, CA, AB, and also stand for the circular measures of those arcs respectively. Suppose Ac and Cb to meet in D; Cb and Ba in E; Ba and Ac in E. Then, if all the sides are *concave* towards the opposite corners,

$A - b = \angle EDF$, $B - c = \angle DEF$, $C - A = \angle DFE$,

so that $A + B + C - (a + b + c) = \pi$. But if any side is *convex* towards its opposite corner, it must be taken as negative in the above general relation, as appears by considering that, if, *e.g.*, BC became convex towards A, each of the angles B, C would be diminished by a. When the radius becomes infinite, the lengths of BC, CA, AB remaining finite, $A + B + C = \pi$.

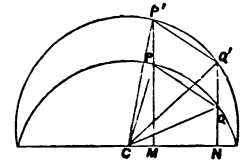


6151. (By G. Turriff, M.A.)—If P, Q be two points on an ellipse such that the difference (α) of their excentric angles is constant, and if the ordinates at P and Q be produced to meet the auxiliary circle in P′ and Q′; show (1) that area of $\Delta P'CQ'$: area of $\Delta PCQ = a : b$, C being the centre; (2) that PQ touches at its middle point the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = \cos^2 \frac{1}{2}\alpha$.

Solution by C. MORGAN, B.A.; J. A. KEALY, M.A.; *and others.*

1. The ellipse is the projection of the auxiliary circle; hence, the triangle PCQ being the projection of P′CQ′, we have

$\Delta PCQ : \Delta CP'Q' = \cos.\text{projective } \angle = b : c.$



2. Let the excentric angles of P, Q be ϕ, $(\phi + \alpha)$, where α is constant; then the coordinates of P, Q are

$[a\cos\phi, b\sin\phi]$, $[a\cos(\phi+\alpha), b\sin(\phi+\alpha)]$;

and the coordinates of the middle point of PQ are

$$x' = a\cos(\phi + \tfrac{1}{2}\alpha)\cos\tfrac{1}{2}\alpha, \quad y' = b\sin(\phi + \tfrac{1}{2}\alpha)\cos\tfrac{1}{2}\alpha;$$

therefore $a^2y'^2 + b^2x'^2 = a^2b^2\cos^2\tfrac{1}{2}\alpha$, *i.e.* (x', y') is on the specified ellipse.

Again, the equation to the line PQ is

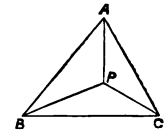
$ay\sin(\phi + \tfrac{1}{2}\alpha) + bx\cos(\phi + \tfrac{1}{2}\alpha) = ab\cos\tfrac{1}{2}\alpha$, or $a^2y'y + b^2x'x = a^2b^2\cos^2\tfrac{1}{2}\alpha$,

which is a tangent to the ellipse specified in the Question.

6399. (By the Rev. H. G. DAY, M.A.)—If (u, v, w) are the distances of a point from the corners A, B, C of a triangle, and (l, m, n) are constants; find the point within the triangle such that $lu + mv + nw$ shall be a minimum, when a minimum is possible.

Solution by Prof. GENESE, M.A.; W. B. GROVE, B.A.; *and others.*

Let P be the required position; then, evidently, if the point P were acted on by three tensions towards A, B, C, proportional to l, m, n, these tensions would mutually balance; hence no minimum is possible unless lines proportional to l, m, n would form a triangle, and the three angles BPC, CPA, APB taken in order will be equal to the exterior angles of this triangle.

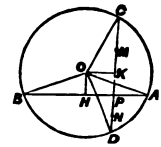


The only other requisite condition is that BPC may be greater than A, &c., or that the sum of any two corresponding angles must be less than two right angles.

All these conditions can be derived from the obvious fact, that a circle with A as centre must touch in P a Cartesian oval $mv + nw =$ constant.

5947. (By Prof. SEITZ, M.A.)—Two points are taken at random within a circle, one on each side of a chord which divides the circle into two segments whose areas are u_1, u_2; show that the chance that the chord drawn through the points is less than a chord which cuts an area u from the circle, is $\dfrac{u^2}{u_1u_2}$, when $u < u_1 < u_2$, and $\dfrac{2u - u_1}{u_2}$, when $u > u_1$.

Solution by the PROPOSER.



Let M, N be two random points on opposite sides of the chord AB, CD the chord through them, and O the centre of the circle. Draw OH and OK perpendicular to AB and CD.

Let $OA = r$, $PM = x$, $MN = y$, $PC = w_1$, $PD = w_2$, $\angle AOH = \alpha$, $\angle COK = \theta$, $\angle HOK = \phi$, and $2\beta =$ the angle formed by radii drawn to the extremities of a chord which cuts off an area, u.

Then we have $w_1 = r(\sin\theta + \cos\alpha \operatorname{cosec}\phi - \cos\theta\cot\phi)$,

$$w_2 = r(\sin\theta - \cos\alpha\operatorname{cosec}\phi + \cos\theta\cot\phi),$$

$$u = r^2(\beta - \sin\beta\cos\beta), \quad \text{and} \quad u_1 = r^2(\alpha - \sin\alpha\cos\alpha).$$

An element of the circle at M is $r\sin\theta\, d\theta\, dx$, and at N it is $d\phi\, y\, dy$.

When $u < u_1$, the limits of θ are 0 and β; of ϕ, $\alpha-\theta$ and $\alpha+\theta$, and doubled; of x, 0 and w_1; and of y, x and $w_2 + x$. Hence, since the whole number of ways the two points can be taken is $u_1 u_2$, we have

$$p = \frac{2}{u_1 u_2}\int_0^\beta\int_{\alpha-\theta}^{\alpha+\theta}\int_0^{w_1}\int_x^{w_2+x} r\sin\theta\, d\theta\, d\phi\, dx\, y\, dy$$

$$= \frac{r}{u_1 u_2}\int_0^\beta\int_{\alpha-\theta}^{\alpha+\theta}\int_0^{w_1}[(w_2+x)^2 - x^2]\sin\theta\, d\theta\, d\phi\, dx$$

$$= \frac{2r^2}{u_1 u_2}\int_0^\beta\int_{\alpha-\theta}^{\alpha+\theta} w_1 w_2 \sin^2\theta\, d\theta\, d\phi = \frac{4r^4}{u_1 u_2}\int_0^\beta(\theta - \sin\theta\cos\theta)\sin^2\theta\, d\theta = \frac{u^2}{u_1 u_2}.$$

When $u > u_1$, the limits of ϕ are $\alpha-\theta$ and $\alpha+\theta$, when $\theta < \alpha$; and $\theta-\alpha$ and $\theta+\alpha$, when $\theta > \alpha$. Hence we have

$$p_1 = \frac{2r^2}{u_1 u_2}\int_0^\alpha\int_{\alpha-\theta}^{\alpha+\theta} w_1 w_2\sin^2\theta\, d\theta\, d\phi + \frac{2r^2}{u_1 u_2}\int_\alpha^\beta\int_{\theta-\alpha}^{\theta+\alpha} w_1 w_2\sin^2\theta\, d\theta\, d\phi$$

$$= \frac{4r^4}{u_1 u_2}\int_0^\alpha(\theta - \sin\theta\cos\theta)\sin^2\theta\, d\theta + \frac{4r^4}{u_1 u_2}\int_\alpha^\beta(\alpha - \sin\alpha\cos\alpha)\sin^2\theta\, d\theta = \frac{2u - u_1}{u_2}.$$

[When $u_1 = u_2 = \frac{1}{2}\pi r^2$, we have $p = \frac{1}{\pi^2}(2\beta - \sin 2\beta)^2$, which is the chance required in Prof. SYLVESTER'S Quest. 1849, whereof two other solutions are given on pp. 92, 93 of Vol. VIII. of our *Reprints*.]

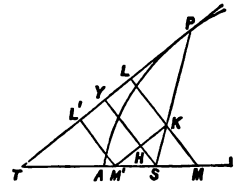
6211. (By R. E. RILEY, B.A.)—Two points are taken on the axis of a parabola at equal distances on opposite sides of the focus, and from these points perpendiculars are drawn to a tangent. If SY be the perpendicular from the focus, show that the difference of the other two perpendiculars varies inversely as SY.

Solution by A. ANDERSON; C. MORGAN, B.A.; J. O'REGAN; *and others.*

Let ML, M'L' be the perpendiculars, and let SM = SM' = c; then we have

$$ML - M'L' = 2c\left(\frac{SY}{ST}\right) = \frac{2c \cdot SY}{SP} \propto \frac{1}{SY},$$

since $SY^2 = SP \cdot SA$.

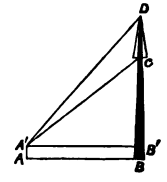


6315. (By Prof. MATZ, M.A.)—The dome of the Capitol at Washington rises m (= 396½ feet) above the level of the ground, and is surmounted by the Statue of Freedom, whose height is n (= 18 feet); show that an observer whose eye is k (= 5½ feet) above the ground, can get the best view of the Statue when his distance from the axis of the dome is

$$\{(m-k)[(m-k)+n]\}^{\frac{1}{2}} = 399{\cdot}8987372 + \text{feet}.$$

Solution by J. A. KEALY, M.A.; E. RUTTER; *and others.*

It is clear that, if a circle be drawn through the highest and lowest points (D, C) of the Statue, so as to touch at A' the line (A'B') through the observer's eye, parallel to the horizon, the visual angle CA'D will afford the observer the best view of the Statue; hence $A'B' = (B'C \cdot B'D)^{\frac{1}{2}} = 399{\cdot}8988425$ feet.



5889. (By J. J. WALKER, M.A.)—Show that $la+m\beta+n\gamma$ will be a side of the cone having origin as vertex, and the circle through the terms of α, β, γ as base, if

$$l^{-1}T^2(\beta-\gamma)+m^{-1}T^2(\gamma-\alpha)+n^{-1}T^2(\alpha-\beta) = 0;$$

and hence deduce immediately the existence of the second system of circular sections parallel to the plane through the terms of α^{-1}, β^{-1}, γ^{-1}.

Solution by G. TURRIFF, M.A.; Prof. EVANS; *and others.*

Let $S\epsilon\rho = -\epsilon^2$ be the equation of the plane through terms of α, β, γ, and $\rho^2 = 2S\delta\rho$ the equation of the sphere through the origin and terms of α, β, γ; hence the equation of the cone is $\rho^2\epsilon^2 + 2S\epsilon\rho S\delta\rho = 0$. The former two equations are satisfied by α, β, γ. Now, if $la+m\beta+n\gamma$ be a line in cone $(la+m\beta+n\gamma)^2\epsilon^2 + 2S\epsilon(la+m\beta+n\gamma)S\delta(la+m\beta+n\gamma) = 0$,

or $2lm\,S\alpha\beta + 2ln\,S\gamma\alpha + 2mn\,S\beta\gamma - (lm+ln)\,\alpha^2 - (ml+mn)\,\beta^2 - (nl+nm)\,\gamma^2 = 0.$

But $lm\,(\alpha-\beta)^2 = lm\alpha^2 + lm\beta^2 - 2lm\,S\alpha\beta$, &c., &c.;

therefore $lm\,(\alpha-\beta)^2 + mn\,(\beta-\gamma)^2 + nl\,(\gamma-\alpha)^2 = 0,$

or $n^{-1}T^2(\alpha-\beta) + l^{-}\ T^2(\beta-\gamma) + m^{-1}T^2(\gamma-\alpha) = 0.$

Again, $l\alpha + m\beta + n\gamma$ may be written $l\alpha^2 . \alpha^{-1} + m\beta^2 . \beta^{-1} + n\gamma^2 . \gamma^{-1}$, and taking $(\alpha^{-1}, \beta^{-1}, \gamma^{-1})$ for (α, β, γ); we should get, in a similar way,

$(n\gamma^2)^{-1}T^2(\alpha^{-1}-\beta^{-1}) + (l\alpha^2)^{-1}T^2(\beta^{-1}-\gamma^{-1}) + (m\beta^2)^{-1}T^2(\gamma^{-1}-\alpha^{-1}) = 0,$

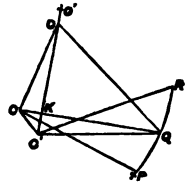
or $$\frac{n^{-1}T^2(\alpha-\beta) + l^{-1}T^2(\beta-\gamma) + m^{-1}T^2(\gamma-\alpha)}{\alpha^2\beta^2\gamma^2} = 0,$$

the same equation as the last; hence, &c.

6212. (By D. Edwardes.)—If ρ, R be the radii of absolute and spherical curvature at an ordinary point on a curve in space, and if ds, ds' be the elements of arcs at corresponding points on the locus of the centres of absolute and spherical curvature respectively, prove that

$$ds : ds' = d\rho : d\mathrm{R}.$$

Solution by G. J. Griffiths, M.A.; J. O'Regan; *and others.*



Let P, Q, R be three consecutive points on a curve; C, C′ the centres of absolute curvature at Q, R; O the centre of spherical curvature—the intersection of the consecutive normal planes OCQ, OC′R; $CQ = \rho$, $OQ = R$, $\angle CQC'$, the angle of torsion, $= dT = \angle OCO'$. Since C, C′ lie on the circle of which OQ is the diameter, $CC' \equiv ds = R\,dT$.

If O′ be the consecutive centre of spherical curvature, we may regard OO′ as an element of a plane curve, to which CO, C′O′ are consecutive tangents, and QC, QC′ perpendiculars on them from the origin Q. Remembering the expression $r\frac{dr}{dp}$, for the radius of curvature, we get

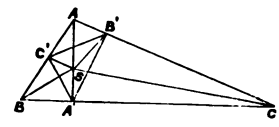
$$\frac{OO'}{\angle COC'} = \mathrm{R}\frac{d\mathrm{R}}{d\rho}, \text{ therefore } ds' = \mathrm{R}\,dT\,\frac{d\mathrm{R}}{d\rho};$$

whence $$ds : ds' = d\rho : d\mathrm{R}.$$

5620. (By E. W. Symons, M.A.)—Prove that the latus rectum of a conic inscribed in a triangle ABC is equal to $8\mathrm{R}\frac{\delta_1\delta_2\delta_3}{d_1d_2d_3}$, where $\delta_1, \delta_2, \delta_3, d_1, d_2, d_3$ are the distances of the sides and vertices from either focus.

Solution by D. WICKERSHAM, B.D.; *the* PROPOSER; *and others.*

Let S be either focus; join SA, SB, SC; and draw SA′, SB′, SC′ perpendicular to the sides; then, if α, β be the semi-axes of the conic, we have $\left(\frac{\beta^2}{\delta_1}, \frac{\beta^2}{\delta_2}, \frac{\beta^2}{\delta_3}\right)$ as the distances of the *other* focus from the sides;



hence $\delta_1 \sin A + \ldots + \ldots = \frac{\beta^2}{\delta_1} \sin A + \ldots + \ldots$; $\therefore\ \beta^2 = \frac{\delta_1\delta_2\delta_3(\delta_1 \sin A + \ldots)}{\delta_2\delta_3 \sin A + \ldots}$.

Now, by the auxiliary circle property of conics, we have

$$\alpha = \text{semi-major axis} = \text{radius of circle through A′B′C′}$$

$$= \frac{B'C' \cdot C'A' \cdot A'B'}{4 \text{ area } A'B'C'} = \frac{d_1 d_2 d_3 \sin A \sin B \sin C}{2(\delta_2\delta_3 \sin A + \ldots)};$$

therefore $\text{latus rectum} = \frac{2\beta^2}{\alpha} = \frac{4\,\delta_1\delta_2\delta_3(\delta_1 \sin A + \ldots)}{d_1 d_2 d_3 \sin A \sin B \sin C}$;

but $\delta_1 \sin A + \ldots = \frac{\delta_1 a + \ldots}{2R} = \frac{2\Delta}{2R} = 2R \sin A \sin B \sin C$;

therefore $l = 8R \frac{\delta_1\delta_2\delta_3}{d_1 d_2 d_3}$.

6238. By R. KNOWLES, B.A., L.C.P.)—A paraboloid, of parameter m, when prevented from sliding down a plane whose inclination is α, just stands on its base without falling over; prove that the length of its axis is $9m \cot^3 \alpha$.

Solution by A. ANDERSON, M.A.; C. BICKERDIKE, L.C.P.; *and others.*

Let h equal height of axis, then, if l equal radius of base, we have

$$l^2 = mh, \quad \tan \alpha = \frac{3l}{h}; \text{ whence } h = 9m \cot^2 \alpha.$$

6266. (By E. W. SYMONS, M.A.)—Prove that the tetrahedron of reference, and the tetrahedron formed by the tangent planes at its vertices to the conicoid $(l+m+n)(lyz+mzx+nxy) = mn\,xw + nl\,yw + lm\,zw$, are such that the lines joining corresponding vertices meet in a point; and that when l, m, n vary, this point moves on the surface

$$x^{-1} + y^{-1} + z^{-1} + w^{-1} = 0.$$

Solution by J. E. A. STEGGALL, B.A.; J. W. RUSSELL, M.A.; *and others.*

Using $l+m+n=-r$, the conicoid is

$$\frac{yz}{mn}+\ldots+\ldots+\frac{xw}{lr}+\ldots,$$

or $\left(\text{if } \frac{x}{e}=X, \text{ etc.}\right)$ $YZ+ZX+XY+XW+WY+WZ=0$;

and the tangent planes at A, B, C, D are

$Y+Z+W=0$, $X+Z+W=0$, $X+Y+W=0$, $X+Y+Z=0$...(1, 2, 3, 4);

and the line through A and the intersection of the planes (2), (3), (4) is

$$Y=Z=W, \text{ that is } \frac{y}{m}=\frac{z}{n}=\frac{w}{r};$$

and similarly for the others; and they clearly meet in the point

$\frac{x}{l}=\frac{y}{m}=\frac{z}{n}=\frac{w}{r}$; therefore $x+y+z+w=0$, if l, m, n vary.

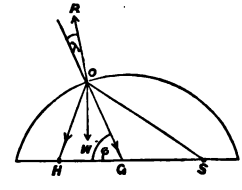
6148. (By D. EDWARDES.)—A ring of weight W is just capable of sliding upon a rough semi-elliptic wire, and has attached to it two strings, which, passing through smooth small rings on the foci, support equal weights (P). If 2θ be the angle between the strings in a limiting position of equilibrium, prove that $W^2\sin^2\theta+\mu^2(W\sin\theta+2Pe\cos\theta)^2=e^2W^2$, where μ is the coefficient of friction, and e eccentricity of ellipse.

Solution by J. L. KITCHIN, M.A.; G. TURRIFF, M.A.; *and others.*

Let O be the position of the ring, and ϕ the inclination to the axis of the normal at O; then, λ being the angle of friction, we have, for the equilibrium of the ring,

$$\frac{W}{\sin\lambda}=\frac{2P\cos\theta}{\cos(\phi+\lambda)},$$

or
$$\frac{W}{\mu}=\frac{2P\cos\theta}{\cos\phi-\mu\sin\phi} \quad \ldots\ldots\ldots (1).$$



The geometrical equation is

$$HG=e\,.\,OH, \quad \text{or} \quad \sin\theta=e\sin\phi \quad \ldots\ldots\ldots (2).$$

Eliminating ϕ between (1) and (2), we have the result in question.

5926. (By Professor SYLVESTER, F.R.S.)—By the potential work (in respect to an arbitrary centre) of a body describing any orbit, plane or twisted,

understand what the work would be were it moving in a circle about such centre under the influence of the forces actually soliciting it; and by the residual work understand the excess of the actual stored-up work over such potential work. Also in general let an acceleration in any quantity (q) signify its second time-derivative $\left(\frac{d^2q}{dt^2}\right)$. Prove that the residual work of a particle of unit mass at each moment of time, in respect to any centre, is equal to the acceleration in the square of the radius vector drawn to such centre. [Prof. SYLVESTER calls this the *equation of residual work*. It is substantially the principle by which Lagrange reduces to quadratures the problem of the motion of a planet about two fixed suns.]

Solution by W. J. CURRAN SHARP, M.A.

Let $\frac{d^2x}{dt^2} = X$, $\frac{d^2y}{dt^2} = Y$, $\frac{d^2z}{dt^2} = Z$ be the equations of motion; then, for motion in a circle, $x\frac{dx}{dt} + y\frac{dy}{dt} + z\frac{dz}{dt} = 0$;

therefore $x\frac{d^2x}{dt^2} + y\frac{d^2y}{dt^2} + z\frac{d^2z}{dt^2} + \left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2 = 0$;

therefore the potential work

$$= -\left(x\frac{d^2x}{dt^2} + y\frac{d^2y}{dt^2} + z\frac{d^2z}{dt^2}\right) = -(xX + yY + zZ),$$

the actual work $= 2\int(Xdx + Ydy + Zdz)$,

and the residual work $= 2\int(Xdx + Ydy + Zdz) + (xX + yY + zZ)$,

$$= x\frac{d^2x}{dt^2} + y\frac{d^2y}{dt^2} + z\frac{d^2z}{dt^2} + \left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2 = \tfrac{1}{2}\frac{d^2(r^2)}{dt^2}.$$

6128. (By Prof. TANNER, M.A.)—The continued fraction $\frac{a}{b-}\frac{a}{b-}\cdots\frac{a}{b-}\frac{a}{b+x}$ (r terms) being represented by u_r, find a, b so that, for all values of r, $u_r = u_{r+n}$, and verify the result by showing the form of u_r.

Solution by Prof. WOLSTENHOLME, M.A.; Prof. PRATT; *and others.*

The p^{th} convergent to the fraction $\frac{a}{b-}\frac{a}{b-}\frac{a}{b-}\ldots$, is $\frac{u_p}{v_p}$, where u_p, v_p are each of the form $\lambda\alpha^{p-1} + \mu\beta^{p-1}$, and α, β are the roots of the equation $z^2 - bz + a = 0$.

For u, we have $\lambda + \mu = a$, $\lambda\alpha + \mu\beta = ab$; therefore

$$\lambda(\beta - \alpha) = a(\beta - b) = -a\alpha, \quad \text{or} \quad \lambda = \frac{a\alpha}{\alpha - \beta}, \quad \mu = \frac{a\beta}{\beta - \alpha}.$$

For v, we have $\lambda+\mu=b$, $\lambda\alpha+\mu\beta=b^2-a$, and $\lambda(\beta-\alpha)=b(\beta-b)+a=a-b\alpha=a(\beta-b)=-a^2$, or $\lambda=\frac{a^2}{a-\beta}$, $\mu=\frac{\beta^2}{\beta-\alpha}$; hence

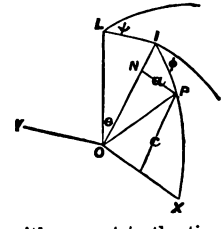
$$U_r=\frac{(b+x)\frac{a(\alpha^{r-1}-\beta^{r-1})}{\alpha-\beta}-a^2.\frac{\alpha^{r-2}-\beta^{r-2}}{\alpha-\beta}}{(b+x)\frac{\alpha^r-\beta^r}{\alpha-\beta}-a.\frac{\alpha^{r-1}-\beta^{r-1}}{\alpha-\beta}}=a.\frac{\alpha^r-\beta^r+x(\alpha^{r-1}-\beta^{r-1})}{\alpha^{r+1}-\beta^{r+1}+x(\alpha^r-\beta^r)};$$

and U_r will be equal to U_{n+r}, if $\alpha^n=\beta^n$, or $\frac{\beta}{\alpha}=\cos\frac{2p\pi}{n}+i\sin\frac{2p\pi}{n}$, $\frac{\beta}{\alpha}+\frac{\alpha}{\beta}=2\cos\frac{2p\pi}{n}=\frac{b^2}{a}-2$, or $b^2=4a\cos^2\frac{p\pi}{n}$, where p is any whole number not a multiple of n. If we write $\frac{p\pi}{n}=\theta$, we may obtain U_r in the form $a^{\frac{1}{2}}\frac{a^{\frac{1}{2}}\sin r\theta+x\sin(r-1)\theta}{a^{\frac{1}{2}}\sin(r+1)\theta+x\sin r\theta}$, and writing $n+r$ for r amounts to adding $p\pi$ to each of the arguments, and therefore leaves the value unchanged.

6341. (By Prof. MINCHIN, M.A.)—A rigid body moves about a fixed point O, the motion of the instantaneous axis OI being completely given, as also the angular velocity of the body at each instant about it; find the components of the acceleration of any particle P, in the body along OI, a perpendicular to OI and OP, and a line perpendicular to these two directions.

Solution by G. S. CARR, B.A.; C. MORGAN, B.A.; *and others.*

Let θ, ψ be polar coordinates of OI referred to a fixed line in space OL and a fixed plane through it; ϕ the angle between the planes POI and IOL; $a=$PN the perpendicular on OI; $c=$ON. Let $\omega_1, \omega_2, \omega_3$ be the component angular velocities of the body about rectangular axes OX, OY, OI; the plane IOX being that of IOP. The question requires the accelerations of P in three fixed directions parallel at the moment to these moving axes.



Since OI is the instantaneous axis, $\omega_1=\omega_2=0$, and ω_3 is given. Also (indicating differentiations with respect to the time by dashes) $\theta'=\psi'=0$, and $\phi'=\omega_3$. The accelerations of P parallel to OX, OY, OI will be $c\omega_2'-a\omega_3^2$, $a\omega_3'-c\omega_1'$, $-a\omega_2'$.

Taking the usual transformations of $\omega_1, \omega_2, \omega_3$ below (ROUTH'S *Rigid Dyn.*, 3rd ed., p. 191), differentiating, and putting $\theta'=\psi'=0$, we have

$$\left.\begin{aligned}\omega_1&=\theta'\sin\phi-\psi'\sin\theta\cos\phi\\ \omega_2&=\theta'\cos\phi+\psi'\sin\theta\sin\phi\\ \omega_3&=\psi'\cos\theta+\phi'\end{aligned}\right\};\quad\therefore\left\{\begin{aligned}\omega_1'&=\theta''\sin\phi-\psi''\sin\theta\cos\phi,\\ \omega_2'&=\theta''\cos\phi+\psi''\sin\theta\sin\phi,\\ \omega_3'&=\psi''\cos\theta+\phi''.\end{aligned}\right.$$

These values of $\omega_1', \omega_2', \omega_3'$, substituted in the above expressions, give the required accelerations in terms of known quantities.

6249. (By Prof. MATZ, M.A.)—Prove that the mean minimum eccentricity of an ellipse capable of resting in equilibrium on a *rough* inclined plane is $e_1 = 2(\sqrt{2}-1)$.

Solution by G. EASTWOOD, M.A.; D. EDWARDES; *and others.*

If (a, b) be the semi-axes of an ellipse, the maximum value of the angle between the normal at any point and the central radius vector is $\cos^{-1}\frac{2ab}{a^2+b^2}$; hence, for equilibrium, if i be the inclination of the plane, we must have $\cos i \nless \frac{2ab}{a^2+b^2}$, whence the minimum value of the excentricity is $\left(\frac{2\sin i}{1+\sin i}\right)^{\frac{1}{2}}$ (MINCHIN'S *Statics*, p. 191, Ex. 15); and the mean value is

$$\frac{2}{\pi}\int_0^{\frac{1}{2}\pi}\left(\frac{2\sin i}{1+\sin i}\right)^{\frac{1}{2}}di = \frac{2}{\pi}\int_0^{\frac{1}{2}\pi}\left(\frac{2\cos i}{1+\cos i}\right)^{\frac{1}{2}}di = \frac{2}{\pi}\int_0^{\frac{1}{2}\pi}(1-\tan^2\tfrac{1}{2}i)^{\frac{1}{2}}di$$

$$= (\text{putting } \tan\tfrac{1}{2}i = \sin\phi)\ \frac{2}{\pi}\int_0^{\frac{1}{2}\pi}(1-\tan^2\tfrac{1}{2}i)^{\frac{1}{2}}di = \frac{4}{\pi}\int_0^{\frac{1}{2}\pi}\frac{\cos^2\phi\,d\phi}{1+\sin^2\phi}$$

$$= 2(\sqrt{2}-1).$$

6407. (By Prof. CROFTON, F.R.S.)—If, in a triangle, $C = 60°$, prove that

$$\frac{1}{a+c}+\frac{1}{b+c} = \frac{3}{a+b+c}.$$

Solution by H. MURPHY; H. L. ORCHARD, M.A.; *and others.*

$$\cos C = \tfrac{1}{2} = \frac{b^2+a^2-c^2}{2ab};\quad ab = b^2+a^2-c^2;$$

$$3ab = (a+b)^2-c^2 = (a+b+c)(a+b-c);$$

therefore

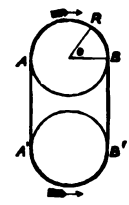
$$\frac{3}{a+b+c} = \frac{a+b-c}{ab} = \frac{1}{a+c}+\frac{1}{b+c}.$$

6416. (By Dr. HOPKINSON, F.R.S.)—An endless slightly extensible strap is stretched over two equal pulleys; prove that the maximum couple which the strap can exert on either pulley is

$$\frac{2a(c+\pi a)}{c\coth\frac{1}{2}\mu\pi+\frac{2a}{\mu}}\,T,$$

where a is the radius of either pulley, c the distance of their centres, μ the coefficient of friction, and T the tension with which the strap is put on.

Solution by G. S. CARR, B.A.; K. S. PUTNAM, M.A.; *and others.*



Let the pulleys be gradually urged in contrary directions, as shown by the arrows, until they slip underneath the strap. The portion AA′ of the strap will lengthen; BB′ will contract. Each element of the strap in succession between A and B beginning at A will slip upon the pulley and then be extended, each element being less extended than the preceding one. When B is about to slip the maximum resistance is reached. Let this be the condition in the figure.

Let P, Q be the tensions, at that moment, of AA′ and BB′. Let the original length of BB′, at the tension T, be $c+x$; that of AA′, $c-y$; and consequently that of the portion AB,

$$\pi a - \tfrac{1}{2}(x-y) \quad \ldots\ldots(1).$$

To find the value of (1); let ds be an element of the strap, at R, at the original tension T; $d\sigma$ the extended length of the same element, and τ the corresponding increased tension. Then, by HOOKE's law,

$$\frac{d\sigma}{ds} = \left(1+\frac{\tau}{\lambda}\right) \div \left(1+\frac{T}{\lambda}\right). \quad \text{Let } 1+\frac{T}{\lambda} = k.$$

The difference between AB and its original length is

$$\pi a - \int_0^{\pi a} \frac{ds}{d\sigma}\, d\sigma = \int_0^{\pi a} \frac{d\sigma}{ds}\, ds - \pi a = \frac{a}{k}\int_0 \left(1+\frac{\tau}{\lambda}\right) d\theta - \pi a \quad \ldots\ldots(2).$$

The *slight extensibility* justifies the first equation, and also the subsequent approximations in this demonstration.

Now, if $\angle BR = \theta$, we know that $\tau = e^{\mu\theta}Q$ and $P = e^{\mu\pi}Q$.....(3).
Therefore, by (1) and (2),

$$x-y = \frac{2a}{k}\int_0^{\pi}\left(1+\frac{Q}{\lambda}e^{\mu\theta}\right)d\theta - \pi a = \frac{2a}{k}\left\{\pi + \frac{Q}{\lambda\mu}(e^{\mu\pi}-1)\right\} - \pi a$$

$$= \frac{2Qa}{\lambda\mu}(e^{\mu\pi}-1) - \frac{2T\pi a}{\lambda}, \text{ by the value of } k.$$

Again, $c+x : c = 1+\frac{T}{\lambda} : 1+\frac{Q}{\lambda}$ and $c-y : c = 1+\frac{T}{\lambda} : 1+\frac{P}{\lambda}$,

$$\therefore\ 1+\frac{x}{c}\left(1+\frac{T}{\lambda}\right)\left(1+\frac{Q}{\lambda}\right)^{-1} = 1+\frac{T-Q}{\lambda}. \quad \text{Similarly } 1-\frac{y}{c} = 1+\frac{T-P}{\lambda},$$

$$\therefore \qquad x-y = \frac{c}{\lambda}(2T-P-Q) = \frac{c}{\lambda}\left\{2T-Q(e^{\mu\pi}+1)\right\} \text{ by (3).}$$

Equating this and the former value of $x-y$, we find the value of Q following. The acting couple is

$$(P-Q)\,a = Q(e^{\mu\pi}-1)\,a = \frac{2a(c+\pi a)T}{c\,\dfrac{e^{\mu\pi}+1}{e^{\mu\pi}-1}+\dfrac{2a}{\mu}}.$$

But $\dfrac{e^{\mu\pi}+1}{e^{\mu\pi}-1} = \dfrac{e^{\frac{1}{2}\mu\pi}+e^{-\frac{1}{2}\mu\pi}}{e^{\frac{1}{2}\mu\pi}-e^{-\frac{1}{2}\mu\pi}} = \coth(\tfrac{1}{2}\mu\pi)$. Hence the result.

6235. (By H. McColl, B.A.)—When, from the three implications, (1) $a'b + ab' : dx$, (2) $ax + by : c$, (3) $cd : y$, may we conclude that either x or y is true, but not both?

Solution by the Proposer.

We are required to find the *weakest antecedent* of $x'y + xy'$; but it will be more convenient to find first the *strongest consequent* of $(x'y + xy')'$, that is, of $xy + x'y'$, and then transpose. Thus, using rule 23 and formula 6, we get $xy : (ab + a'b' + d)(a'b' + c)$, $x'y' : (ab + a'b')(c' + d')$.

Hence $xy + x'y' : ab + a'b' + cd$, omitting redundant terms.

Transposing (*i.e.*, by contra-position), we get

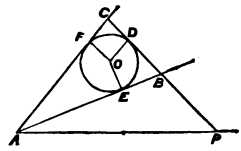
$$a'bc' + a'bd' + ab'c' + ab'd' : x'y + xy'.$$

Using rule 23 and formula 6 for the discovery of zero terms in the antecedent of this implication, we get $a'bc' : dxy'$, $a'bd' : 0$, $ab'c' : 0$, $ab'd' : 0$; hence, finally, $a'bc' : x'y + xy'$.

5086. (By the Editor.)—Through a given point P draw a straight line cutting in B and C the sides of a given angle A, in such wise that AB and AC may be together equal to BC and a given line.

Solution by J. O'Regan; E. Rutter; *and others.*

On AB and AC take AE and AF each equal to ½M (M being the given line); at E and F draw perpendiculars to AB, AC meeting in O; then a circle drawn with O as centre and E as radius will touch AB and AC at E and F; from P draw PBC touching this circle at D, then PBC will be the line required.



For $AF + CF + AE + BE = AF + AE + BC$,

or $AB + AC = 2AE + BC = M + BC$.

5976. (By R. Tucker, M.A.) — P_1, P_2, P_3; (Q_1, Q'_1), (Q_2, Q'_2), (Q_3, Q'_3) are the respective denominators and numerators of the ultimate (proper fraction) convergents, which are obtained from making the roots of the equation $x^3 + qx + r = 0$ the quotients of a series of continued fractions; shew that

$$\frac{Q_1 Q'_1}{P_1} + \frac{Q_2 Q'_2}{P_2} + \frac{Q_3 Q'_3}{P_3} = \frac{1}{P_1} + \frac{1}{P_2} + \frac{1}{P_3}.$$

Solution by the PROPOSER; J. O'REGAN; *and others.*

Let a, b, c be the roots of the equation; then we have

$$\frac{Q_1}{P_1} = \frac{1}{b+a+\frac{1}{c}} = \frac{ac+1}{abc+b+c}, \quad \frac{Q'_1}{P_1} = \frac{ab+1}{abc+b+c};$$

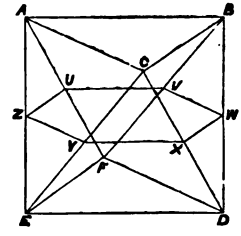
therefore $$\frac{Q_1 Q'_1 - 1}{P_1} = a, \text{ \&c.};$$

hence $$\frac{Q_1 Q'_1 - 1}{P_1} + \frac{Q_2 Q'_2 - 1}{P_2} + \frac{Q_3 Q'_3 - 1}{P_3} = a + b + c = 0.$$

6420. (By DONALD MCALISTER, D.Sc.)—A regular octohedron is cut by a plane parallel to one of its faces; prove that the perimeter of the section is constant.

Solution by Professor TOWNSEND, F.R.S.

Let ABCDEF (see fig.) be the six vertices of the solid, and UVWXYZ those of its hexagonal section by any plane parallel to its pair of opposite faces ABC and DEF; then, on account of the parallelism of the three tetrads of lines (AB, DE, UV, XY), (BC, EF, WX, ZU), (CA, FD, YZ, VW), and on account of the equilateralism of the eight triangular faces of the solid, we have at once, UV = VF and VW = VB, WX = XD and XY = XC, YZ = ZE and ZU = ZA; therefore, at once, perimeter of section = BF + CD + EA, = perimeter of any face of solid; and therefore, &c.



The plane of section being supposed to vary, and to assume consecutively all positions intermediate to ABC and DEF; then, on account of the three squares ABCD, BCEF, CAFD, whose planes intersect at right angles at the centre, and pass through the twelve edges of the solid, the three diagonals UX, VY, WZ of the varying section continue constantly parallel and equal to the three pairs of corresponding sides CA and FD, BC and EF, AB and DE of the two equilateral triangles ABC and DEF; and the equilateral triangle of varying magnitude they determine in every position is the intersection of their varying plane with the aforesaid triad of fixed planes intersecting at the centre and passing through the twelve edges of the solid.

6106. (By A. MARTIN, M.A.)—Show that the mean value of all the radius-vectors that can be drawn from one end of the major axis of a given ellipse to its circumference is $\frac{4b}{\pi e} \sin^{-1} e$.

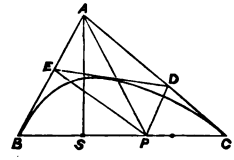
Solution by D. EDWARDES; J. A. KEALY, M.A.; *and others.*

The polar equation being $r = \frac{2ab^2 \cos\theta}{b^2 + a^2e^2 \sin^2\theta}$, the required average is

$$\frac{\int r\,d\theta}{\int d\theta} = \frac{2ab^2}{\pi} \int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi} \frac{d\sin\theta}{b^2 + a^2e^2\sin^2\theta} = \frac{4b}{\pi e} \sin^{-1} e.$$

6402. (By W. E. WRIGHT, B.A.) — Find the envelop of the line joining the feet of the perpendiculars drawn on the sides of a right-angled triangle from any point in the hypotenuse.

Solution by C. MORGAN, B.A.; G. TURRIFF, M.A.; *and others.*



Let P be any point in BC; PD, PE perpendiculars on AC, AB; and AS a perpendicular to BC; then the circle on PA passes through E, D, S; and, similarly, if other points P′, &c. be taken, the circle on P′A will pass through E′, D′, S, &c.

Now S is a fixed point; hence DE will always touch a parabola whose focus is S; AB, AC being a pair of tangents; since in a parabola the circle through the intersection of three tangents passes also through the focus.

6034. (By Professor SYLVESTER, F.R.S.)—

$U_1, U_2, \ldots U_i$
$V_1, V_2, \ldots V_j$
$W_1, W_2, \ldots W_k$
$\ldots \ldots \ldots \ldots$
$\ldots \ldots \ldots \ldots$

} are homogeneous numerical linear functions of n variables; and $i+j+k+\ldots = n$. Any i homogeneous linear functions, in which the coefficients are all integer, are taken of the U's, j of the V's, k of the W's, and so on. Let R be the resultant of these n new linear functions of the n variables. Investigate a rule for determining the greatest common measure of the infinite number of values of R that can be thus obtained. [Prof. SYLVESTER remarks that this greatest common measure is entitled to be called the Resultant of the given functions taken *in sets*.]

Solution by W. J. C. SHARP, M.A.

Since the various new functions must all vanish, if the given functions do so, R must contain the eliminant of these as a factor; and as this is the only condition which will make all the new functions vanish, it is the only factor which is common to all the values of R, and is therefore the G.C.M. required.

6313. (By Prof. GENESE, M.A.) — Prove that, with the usual notation, the distance from the centre of the inscribed circle of a triangle to the orthocentre is $(2r^2 - 4R^2 \cos A \cos B \cos C)^{\frac{1}{2}}$.

Solution by R. KNOWLES, B.A., L.C.P.; E. RUTTER; *and others.*

Employing trilinear coordinates, we have, for the centre of the inscribed circle, $\alpha_1 = \beta_1 = \gamma_1 = r = \Delta s^{-1}$; and for the orthocentre,

$$\alpha_2 = \frac{2\Delta \cdot \cos B \cdot \cos C}{c \cdot \sin A \cdot \sin B}, \quad \beta_2 = \frac{2\Delta \cdot \cos A \cdot \cos C}{c \cdot \sin A \cdot \sin B}, \quad \gamma_2 = \frac{2\Delta \cdot \cos A \cdot \cos B}{c \cdot \sin A \cdot \sin B};$$

$$\therefore d^2 = \frac{abc}{4\Delta^2}\left\{a \cos A (\alpha_1 - \alpha_2)^2 + b \cos B (\beta_1 - \beta_2)^2 + c \cos C (\gamma_1 - \gamma_2)^2\right\}$$

$$= \frac{abc}{4s^2}(a \cos A + b \cos B + c \cos C) - \frac{ab \cos A \cos B \cos C}{\sin A \sin B}$$

$$= \frac{abcr^2}{4\Delta^2} \cdot \frac{2\Delta}{R} - \frac{a}{\sin A}\,\frac{b}{\sin B} \cos A \cos B \cos C$$

$$= \frac{abcr^2}{bc \sin A} \times \frac{2 \sin A}{a} - 4R^2 \cos A \cos B \cos C$$

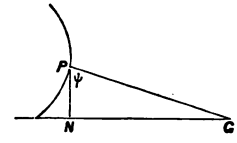
$$= 2r^2 - 4R^2 \cos A \cos B \cos C.$$

5962. (Py Professor COCHEZ.)—Trouver la courbe dont le rapport du rayon de courbure à la normale est constant et égal à m.

Solution by J. HAMMOND, M.A.; G. HEPPEL, M.A.; *and others.*

We have $\rho = \left[1 + \left(\frac{dy}{dx}\right)^2\right]^{\frac{3}{2}} \div \frac{d^2y}{dx^2}$,

$$PG = y \sec \psi = y\left[1 + \left(\frac{dy}{dx}\right)^2\right]^{\frac{1}{2}}.$$



Hence, if $\rho = m$PG, we have

$$\frac{d^2y}{dx^2} \div \left[1 + \left(\frac{dy}{dx}\right)^2\right] = \frac{1}{my}.$$

Multiplying by $2\frac{dy}{dx}$, and integrating, $\log\left[1 + \left(\frac{dy}{dx}\right)^2\right] = \frac{2}{m}\log y + \text{const.}$;

or, if c be a max. or min. value of y, $1 + \left(\frac{dy}{dx}\right)^2 = \left(\frac{y}{c}\right)^{\frac{2}{m}}$ .....(1).

Hence $x = \int dy \div \left[\left(\frac{y}{c}\right)^{\frac{2}{m}} - 1\right]^{\frac{1}{2}}$ is the equation to the curve.

This equation can only be expressed in finite terms for special values of m. Equation (1) can be put in the form $y = c \sec^m \psi$.....(2).

When $m = 1$ the curve is the common catenary, and when $m = -1$ the circle, when $m = \pm \frac{1}{3}$ the integral equation contains an expression of the 4th degree under the radical, and when $m = \pm \frac{2}{3}$ a cubic expression; in these cases the relation between x and y is capable of being expressed by means of the Elliptic Functions.

The case $m = 2$ gives a parabola, whose directrix is the axis of x.

6279. (By Prof. TOWNSEND, F.R.S.)—An annular lamina of uniform density, bounded internally and externally by concentric circles of any radii, being supposed to attract, according to the law of the inverse cube of the distance, a material particle situated anywhere on the concentric sphere, whose radius is the geometric mean between those of its bounding circles; shew, by any method, that the entire resultant attraction is perpendicular to its plane.

I. *Solution by* A. L. SELBY, M.A.; J. W. SHARPE, M.A.; *and others.*

If $(h, 0, l)$ be the attracted point, the potential is

$$mf \int_b^a \int_0^\pi \frac{r\,dr\,d\theta}{h^2 + l^2 + r^2 - 2hr\cos\theta} = mf\pi \int_b^a \frac{r\,dr}{\left\{(h^2+l^2)^2 + 2(l^2-h^2)r^2 + r^4\right\}^{\frac{1}{2}}}$$

$$= mf\frac{\pi}{2} \log \frac{l^2 - h^2 + a^2 + \left\{(h^2+l^2)^2 + 2(l^2-h^2)a^2 + a^4\right\}^{\frac{1}{2}}}{l^2 - h^2 + b^2 + \left\{(h^2+l^2)^2 + 2(l^2-h^2)b^2 + b^4\right\}^{\frac{1}{2}}};$$

$$\therefore \mathrm{X} = \frac{d}{ah}(\text{potential}) = mf\pi \left\{ \frac{-h + \dfrac{h(h^2+l^2) - ha^2}{\left\{(h^2+l^2)^2 + 2(l^2-h^2)a^2 + a^4\right\}^{\frac{1}{2}}}}{l^2 - h^2 + a^2 + \left\{(h^2+l^2)^2 + 2(l^2-h^2)a^2 + a^4\right\}^{\frac{1}{2}}} - \frac{-h + \dfrac{h(h^2+l^2)^2 - h^2}{\left\{(h^2+l^2)^2 + 2(l^2-h^2)b^2 + b^4\right\}^{\frac{1}{2}}}}{l^2 - h^2 + a^2 + \left\{(h^2+l^2)^2 + 2(l^2-h^2)b^2 + b^4\right\}^{\frac{1}{2}}} \right\},$$

which vanishes when $h^2 + l^2 = ab$, *i.e.*, when the attracted point is on the concentric sphere, &c.

II. *Solution by the* PROPOSER.

Denoting by ρ the mass per unit of area of the lamina, by a and b the radii of its two bounding circles, by r and s those of any two others in its interior concentric with its mass, for which $rs = ab$, and consequently inverse to each other with respect to the sphere, by $r \pm dr$ and $s \mp ds$ those of a similarly related consecutive pair in either direction, by r_1 and r_2 and by s_1 and s_2 the greatest and least distances of any point P on the sphere from the circumferences of the two circles r and s, by x the perpendicular distance of P from the axis of the lamina, by μ the mass of the attracted particle supposed situated at P, and by f the absolute intensity of the attraction. Then, since (*Reprint*, Vol. XXXII., page 63), the attractions, parallel to the plane of the lamina, of the two slender circular rings

of its mass intercepted between the two pairs of consecutive circles, r and $r \pm dr$, s and $s \mp ds$, are equal respectively to

$$2\pi\mu f\rho \cdot \frac{r\,dr}{r_1^2 r_2^2} \cdot x \quad \text{and} \quad 2\pi\mu f\rho \cdot \frac{s\,ds}{s_1^2 s_2^2} \cdot x,$$

and act in opposite directions, the two spherical segments connecting P with the circumferences of r and s being necessarily one greater and the other less than a hemisphere; and since, by hypothesis, $dr : ds = r : s$, to prove the property for the joint attraction of the two elementary rings, and therefore for that of the two entire segments of the lamina interior and exterior to the sphere, it is consequently only necessary to show that $r_1 r_2 : s_1 s_2 = r : s$, which, as r and s are inverse circles with respect to the sphere, follows at once from a well-known elementary property of any pair of inverse points with respect to a circle.

That the attraction of the lamina, for the same law of force, is evanescent for every element of its mass situated on its circle of intersection with the sphere (*Reprint*, Vol. XXXII., page 95), is, of course, an immediate consequence from the above.

6353. (See p. 51). II. *Solution by* Rev. R. HARLEY, F.R.S.

In BOOLE's notation the four given conditions may be written thus:—

$$x = x(yz + y'z'), \quad wx = wx(yz + y'z'), \quad w'xy = w'xyz, \quad w'xy'z = 0;$$

or, $x(yz + y'z') = 0$, $wx(yz' + y'z) = 0$, $w'xyz' = 0$, $w'xy'z = 0$;

whence, adding the equations together, and remembering that $w + w' = 1$, and $yz + y'z + yz' + y'z' = 1$, we have $x = 0$.

6204. (By J. J. WALKER, M.A.)—Referring to Quest. 5680, show that the conditions for the double contact of two conics—not expressible in terms of the invariants of the system—may be obtained in terms of coefficients of the three contravariant conics of the system.

I. *Solution by* Professor WOLSTENHOLME, M.A.

The equations of S, S′, two conics having double contact, may be written (in an infinite number of ways) in the forms

$$lx^2 + my^2 + nz^2 = 0, \quad lx^2 + my^2 + n'z^2 = 0.$$

Hence the discriminant of $kS + S'$ is $(k+1)^2(kn + n')lm$, which has two equal roots, and it is manifest that this is the only relation between the

roots for all forms of the equations. But the discriminant will have two equal roots when the conics have single contact. Hence the conditions for double contact cannot be expressed in terms of Δ, Θ, Θ', Δ'. The conditions may be found from the fact that $kS+S'$ is a perfect square for that value of k supplied by the two equal roots. Hence, taking the general equations $(abcfgh\Sigma x, y, z)^2 = 0$, $(a'b'c'f'g'h'\Sigma x, y, z)^2 = 0$, we have

$$(ka+a')(kf+f') = (kg+g')(kh+h'),$$

$$(kb+b')(kg+g') = (kh+h')(kf+f'), \quad (kc+c')(kh+h') = (kf+f')(kg+g');$$

which, eliminating k, will give two conditions which must include the one relation between the invariants

$$(\Theta^2-3\Delta\Theta')(\Theta'^2-3\Delta'\Theta)+4(\Theta\Theta'-9\Delta\Delta')^2 = 0.$$

I must own I do not see the *à priori* reason why the two conditions cannot be expressed in terms of the invariants, and I asked the question in the Senate-House Examinations for 1870; but this was the one question left unanswered. It is obvious that the equations given in Vol. XXXI, p. 84, are not *homogeneous* in the proper sense.

Any equation between Δ, Θ, Θ', Δ', expressing a projective relation between the two conics, will be unaltered if we write $\lambda^3\Delta$, $\lambda^2\mu\Theta$, $\lambda\mu^2\Theta'$, $\mu^3\Delta'$ for Δ, Θ, Θ', Δ'. I have never succeeded in obtaining the true conditions in a form which I thought worthy of publication.

[The solution of Quest. 5680, given on p. 86 of Vol. XXXI. of our *Reprints*, is erroneous, in assuming $S-S'$ to be a line-pair.]

II. *Solution by the* PROPOSER.

The equations in k of Prof. WOLSTENHOLME's Solution are, in effect,

$$Fk^2+F_1k+F' = 0, \quad Gk^2+G_1k+G' = 0, \quad Hk^2+H_1k+H' = 0,$$

where $F = gh-a'f$, $F' = g'h'-a'f'$, $F_1 = gh'+g'h-af'-a'f$.....;

and the conditions that these equations shall hold simultaneously for a common value of k, form a solution of the Question proposed, *e.g.*, the condition that the first two should have a common root, combined with the condition that it should satisfy the third. I quite assent to Prof. WOLSTENHOLME's remark as to the absence of any *à priori* reason, &c.

6378. (By Prof. WOLSTENHOLME, M.A.)—Prove that, if $n > 1$,

$$\int_0^1 (1-x)^{n-2}\,dx \int_0^{\frac{1}{2}\pi} \frac{d\theta}{(1-x^2\sin^2\theta)^{\frac{1}{2}n}} = \tfrac{1}{2}\pi \left\{ \frac{\Gamma\frac{1}{2}(n-1)}{\Gamma\frac{1}{2}(n)} \right\}^2.$$

Solution by H. STABENOW, M.A.; A. L. SELBY, B.A.; *and others.*

From the identity

$$(1-x)^{-\kappa} = 1+\kappa x+\frac{\kappa(\kappa+1)}{1.2}x^2+\ldots+\frac{\Gamma(\kappa+m)}{\Gamma(\kappa)\,\Gamma(m+1)}x^m+\ldots = \Sigma A_{\kappa,m}x^m$$

of the curve intercepted between the two pairs of consecutive circles, [illegible] are equal respectively to

$$\text{Imply.}\frac{ab'}{\sqrt{ab}}\sigma \text{ and } \text{Imply.}\frac{ab}{\sqrt{ab'}}\sigma,$$

and in opposite directions. As the two spherical segments containing P with the circumferences C and C' being necessarily one greater and the other less than a hemisphere, and since, by hypothesis, $ab' : ab = \sigma : \alpha$, to prove the property for the joint attraction of the two elementary rings, and therefore for that of the two entire segments of the lemina interior and exterior to the sphere, it is consequently only necessary to show that $\sigma\sqrt{ab} = \alpha\sqrt{ab'}$, which, as a and a' are inverse circles with respect to the sphere, follows at once from a well-known elementary property of any pair of inverse points with respect to a circle.

That the attraction of the lemina, for the same law of force, is even-normal for every element of its mass situated on its circle of intersection with the sphere (Thomson, Vol. XXXIII., page 36), is, of course, an immediate consequence from the above.

6282. (Set p. 32). II. Solution by Rev. R. Harley, F.R.S.

In Boole's notation the four given conditions may be written thus:—

$$z = z\{px+y(z')\},\quad wx = wx\{px+y(z')\},\quad w'xy = w'xyz,\quad w'xy'z = 0;$$

or, $z\{px+y(z')\} = 0$, $wx\{px'+y(z)\} = 0$, $w'xyz' = 0$, $w'xy'z = 0$;

whence, adding the equations together, and remembering that $w + w' = 1$, and $px+y(z)+px'+y(z') = 1$, we have $x = 0$.

6704. (By J. J. Walker, M.A.)—Referring to Quest. 3880, show that the conditions for the double contact of two conics—not expressible in terms of the invariants of the system—may be obtained in terms of the coefficients of the three contravariant conics of the system.

I. *Solution by* Professor Wolstenholme, [illegible]

The equations of S, S′, two conics having double [illegible] (in an infinite number of ways) in the forms

$$lx^2 + my^2 + nz^2 = 0,\quad lx^2 + [illegible]$$

Hence the discriminant of $kS + S'$ is [illegible] equal roots, and it is manifest that t[illegible]

we have

$$U = \int_0^1 (1-x)^{n-2}\,dx \int_0^{\frac{1}{2}\pi} \frac{d\theta}{(1-x^2\sin^2\theta)^{\frac{1}{2}n}}$$

$$= \Sigma \int_0^1 A_{\frac{1}{2}n,m}(1-x)^{n-2}x^{2m}\,dx \int_0^{\frac{1}{2}\pi} \sin^{2m}\theta\,d\theta$$

$$= \Sigma \int_0^1 A_{\frac{1}{2}n,m} \left\{ \frac{1}{2}\,\frac{\Gamma(\frac{1}{2}+m)\,\Gamma(\frac{1}{2})}{\Gamma(m+1)} \right\} (1-x)^{n-2}x^{2m}\,dx$$

$$= \tfrac{1}{2}\pi\,\Sigma \left\{ A_{\frac{1}{2}n,m}\,A_{\frac{1}{2},m}\,\frac{\Gamma(n-1)\,\Gamma(2m+1)}{\Gamma(n+2m)} \right\}.$$

Now $A_{\frac{1}{2}n,m}\,\dfrac{\Gamma(n-1)\,\Gamma(2m+1)}{\Gamma(n+2m)} = \dfrac{\frac{1}{2}n.\frac{1}{2}n+1.\frac{1}{2}n+2\ldots\frac{1}{2}n+m-1}{1.2.3\ldots m}$

$$\times \frac{1.2.3\ldots 2m}{n-1.n.n+1\ldots n+2m-1} = \frac{1.3.5\ldots 2m-1.n.n+2\ldots n+2m-2}{n-1.n.n+1.n+2\ldots n+2m-1}$$

$$= \frac{1.3.5\ldots 2m-1}{n-1.n+1.n+3\ldots n+2m-1} = \frac{1}{2}\,\frac{\Gamma[\frac{1}{2}(n-1)]\,\Gamma(m+\frac{1}{2})}{\Gamma[\frac{1}{2}(n+1)+m]\,\Gamma(\frac{1}{2})};$$

therefore $U = \tfrac{1}{4}\pi\,\Sigma \left\{ A_{\frac{1}{2},m}\,\dfrac{\Gamma[\frac{1}{2}(n-1)]\,\Gamma(m+\frac{1}{2})}{\Gamma[\frac{1}{2}(n+1)+m]\,\Gamma(\frac{1}{2})} \right\}$

$$= \tfrac{1}{4}\pi\,\frac{\Gamma[\frac{1}{2}(n-1)]}{\Gamma(\frac{1}{2}n)}\,\Sigma \int_0^1 \frac{(1-x)^{\frac{1}{2}n-1}x^{m-\frac{1}{2}}}{\Gamma(\frac{1}{2})}\,A_{\frac{1}{2},m}\,dx$$

$$= \tfrac{1}{4}\pi\,\frac{\Gamma[\frac{1}{2}(n-1)]}{\Gamma(\frac{1}{2}n)} \int_0^1 \frac{(1-x)^{\frac{1}{2}(n-1)-1}x^{-\frac{1}{2}}}{\Gamma(\frac{1}{2})}\,dx = \tfrac{1}{4}\pi \left\{ \frac{\Gamma[\frac{1}{2}(n-1)]}{\Gamma(\frac{1}{2}n)} \right\}^2.$$

6293. (By J. W. Russell, M.A.)—Two bags A and B contain each two balls, each of which is either white or black; a ball is drawn from A and put into B, and then a ball is drawn from B and put into A; in both cases the ball turns out to be white; and this double operation is repeated n times; find the probability of its not failing in the $(n+1)^{\text{th}}$ attempt.

Solution by C. J. Monro, M.A.; J. E. W. Steggall, M.A.; *and others.*

It will, I think, be according to the Proposer's intention, if I lay especial stress on the relation of this problem to first principles, by estimating how much assumption is made at each point. Each double operation, resulting as given, leaves the distribution unaltered. Let U_x, V_x, W_x, for A, and u_x, v_x, w_x, for B, be the chances of two white balls, one, none, after the x^{th} operation. Then the answer,—call it P_n,—is

$$(U_n + \tfrac{1}{2}V_n + 0W_n)(u_n + \tfrac{2}{3}v_n + \tfrac{1}{3}w_n),$$

or

$$\frac{1}{2.3}(2U_n + V_n + 0W_n)(3u_n + 2v_n + w_n).$$

In general, also, if $S^{-1}(a, b, \ldots)$ stand for $a, b, \ldots$ divided by their sum, then

$$U_x, V_x, W_x = S^{-1}(2U_{x-1}, V_{x-1}, 0W_{x-1}),$$

and

$$u_x, v_x, w_x = S^{-1}(3u_{x-1}, 2v_{x-1}, w_{x-1}).$$

But this may conceivably not hold when $x = 1$. For the first stage of the

first operation (the only one, as given, which has an unprecedented result) tells us that there is at least one white ball, and this might affect our judgment on the colours of the rest. U_1, &c. would then be arbitrary; but in any case U_x, &c. are $S^{-1}(2^{x-1}U_1, V_1, 0^{x-1}W_1)$, $S^{-1}(3^{x-1}u_1, 2^{x-1}v_1, w_1)$.

It is natural to assume that the colour of one ball is no evidence of another's, and thus U_1, &c., given in terms of U_0, &c.; also that $W_0 = U_0$, $w_0 = u_0$. More arbitrary are $V_0 : U_0$, $v_0 : u_0$; two hypothetical values are obvious, 1 and 2.* We have in these cases, for P_n,

$$\frac{1}{2\,.\,3}\cdot\frac{2^{n+1}+1+0^{n+1}}{2^n+1+0^n}\cdot\frac{3^{n+1}+2^{n+1}+1}{3^n+2^n+1},$$

$$\frac{1}{2\,.\,3}\cdot\frac{2^{n+1}+2+0^{n+1}}{2^n+2+0^n}\cdot\frac{3^{n+1}+2^{n+2}+1}{3^n+2^{n+1}+1}.$$

$P_0 = \frac{1}{3}$ in each case; for, when $n = 0$, $0^n = 1$, n being the number of times W_0 is multiplied by 0. When $n > 0$, we may write for P_n

$$\frac{1}{2\,.\,3}\cdot\frac{2^{n+1}+1}{2^n+1}\cdot\frac{3^{n+1}+2^{n+1}+1}{3^n+2^n+1},\quad \frac{1}{2\,.\,3}\cdot\frac{2^n+1}{2^{n-1}+1}\cdot\frac{3^{n+1}+2^{n+2}+1}{3^n+2^{n+1}+1}.$$

[* See *Reprint*, Vol. XXXI., pp. 100—103, and XXXIII., pp. 107—110].

II. *Solution by the* Proposer.

The contents of the bags A and B at first are

(ww) (ww) (ww) (wb) (wb) (wb) ..... (A),

(ww) (wb) (bb) (ww) (wb) (bb) ..... (B).

The chances of drawing white from A are

$$1,\ 1,\ 1,\ \tfrac{1}{2},\ \tfrac{1}{2},\ \tfrac{1}{2};$$

hence probabilities of the six hypotheses are

$$\tfrac{2}{9},\ \tfrac{2}{9},\ \tfrac{2}{9},\ \tfrac{1}{9},\ \tfrac{1}{9},\ \tfrac{1}{9}.$$

The state of the second bag is now

$$(www)\ (wbw)\ (bbw)\ (www)\ (wbw)\ (bbw),$$

and the chances of drawing a white from B are

$$1,\ \tfrac{2}{3},\ \tfrac{1}{3},\ 1,\ \tfrac{2}{3},\ \tfrac{1}{2}.$$

Hence, probabilities of the six hypotheses as to contents of B are

$$\tfrac{2}{12},\ \tfrac{2}{12},\ \tfrac{1}{12},\ \tfrac{3}{12},\ \tfrac{2}{12},\ \tfrac{1}{12}.$$

Similarly, it is easily seen that the probabilities of the above twelve hypotheses concerning the state of the bags are, after the nth double event, in the ratios of $2^n : 2^n : 2^n : 1 : 1 : 1$,

and of $3^n : 2^n : 1 : 3^n : 2^n : 1$.

The probability of the happening of the event the $(n+1)^{\text{th}}$ time, is therefore the product of $\dfrac{2^n}{3\,(2^n+1)} \times 3 + \dfrac{1}{3\,(2^n+1)} \times \dfrac{3}{2}$

by $\dfrac{3^n}{2\,(3^n+2^n+1)} \times 2 + \dfrac{2^n}{2\,(3^n+2^n+1)} \times \dfrac{4}{3} + \dfrac{1}{2\,(3^n+2^n+1)} \times \dfrac{2}{3}$;

i.e., the required chance is $\dfrac{2^n+\frac{1}{3}}{2^n+1}\cdot\dfrac{3^n+\frac{1}{3}\,.\,2^{n+1}+\frac{1}{3}}{3^n+2^n+1}$,

which is equal to 1 (as it should be) when $n = \infty$.

6438. (By Prof. CAYLEY, F.R.S.)—Find, at any point of a plane curve, the angle between the normal and the line drawn from the point to the centre of the chord parallel and indefinitely near to the tangent at the point; and examine whether a like question applies to a point on a surface and the indicatrix section at such point.

Solution by C. J. MONRO, M.A.

Upon the curve are taken *four* points, though they ultimately coincide. Through them a quadric may be drawn with one degree of freedom. The straight line in question is therefore the diameter of any such quadric in the ultimate position; and this (as is also otherwise evident) must be the same in direction for all quadrics of the system, including that of *five*-point contact.

$V = 0$ being the equation of the curve, and x, y, $x+\xi$, $y+\eta$ the coordinates of the given and of any other point; let A, B, a, h, b be the values of $\frac{dV}{dx}, \ldots \frac{d^2V}{dy^2}$; then the equation of this quadric is

$$A\xi + B\eta + \tfrac{1}{2}a\xi^2 + h\xi\eta + \tfrac{1}{2}b\eta^2 = 0;$$

therefore the equation of the diameter is

$$B(a\xi + h\eta + A) = A(h\xi + b\eta + B);$$

and therefore the tangent of its inclination to the normal is

$$\frac{Ab - Bh}{Ba - Ah}, \quad \text{which, if } \frac{dV}{dx} = 0, \quad \text{is } -\frac{h}{a}.$$

The case of the plane curve is clearer than that of the surface, because the chord is always determined by two points; the choice of these being unambiguous, because the parallels may be as near each other as we please: whereas the section of the surface by a plane parallel to the tangent plane is in general of a higher degree than the indicatrix, and there is something arbitrary in adopting any definition of its centre. We cannot, without showing cause, treat it as a quadric by neglecting indefinitely small quantities, because the distance of the centre of such quadric from the normal, upon which the quantities we should seek depend, is itself infinitely small in comparison with the axes.

But let us take any five points of the section: whatever there is arbitrary in their distribution will presently disappear. A quadric surface through them which touches the given surface at the given point will be determined with one degree of freedom, and the direction of its diameter from this point will be completely determined by the centre of the quadric section through the five points. When the parallel planes coincide, the system of quadric surfaces includes that of closest, or nine-point, contact; therefore the ultimate direction of the diameter is independent of any distribution of the five points about the section of the given surface.

Perhaps, then, this is the problem suggested in the latter part of the question. Find, at any point of a surface, the direction of a line drawn from the point to the centre of a conic described through five points of the section made by a plane parallel and indefinitely near to the tangent plane at the point. This problem is determinate, being reduced to that of finding for the given point the direction of the diameter of the quadric of nine-point contact.

[Prof. CAYLEY states that the intended answer was, that the like question did *not* apply to the case of the surface, and adds that the answer here given is ingenious, and possibly correct, but that he does not know if there is any definite meaning to be put on the expression "th quadric of nine-point contact."]

5933. (By Prof. SYLVESTER, F.R.S.)—The i^{th} involute to a circle being defined as a cyclode of the i^{th} order, and a symmetrical cyclode as one which can be divided into two precisely equal and similar parts; prove that there are as many distinct species of symmetrical cyclodes of the order $2n$ possessing the property that the length of the arc of any one of them is connected by an algebraical equation with the length of the corresponding arc of its pedal, as there are ways of breaking up the number n into parts all unequal to one another.

Solution by W. J. CURRAN SHARP, M.A.

I have shown (*Messenger of Mathematics* for October, 1879) that the equation to the i^{th} evolute of a circle is represented by the intrinsic equation

$$s = a_0\phi^{i+1} + a_1\phi^i + \ldots + a_{i+1}.$$

For a symmetrical cyclode $\pm\phi$ must give the same values of s, and therefore

$$a_1 = a_3 = \ldots = a_{2p+1} = 0;$$

and the equation to a symmetrical cyclode of the $2n^{th}$ order will be

$$s = a_0\phi^{2n+1} + a_2\phi^{2n+1} + \ldots + a_{2n}\phi \equiv \phi\,(a_0a_2 \ldots a_{2n})\,(\phi^2, 1)^n,$$

of which there will be as many distinct species, differentiated by the nature of the initial point, as there are ways of making up n, by the addition of unequal numbers.

p, the perpendicular from centre of circle upon the tangent,

$$= \rho + \rho_2 + \rho_4 + \&c. = \frac{ds}{d\phi} + \frac{d^3s}{d\phi^3} + \&c.;$$

and if σ be the arc of the pedal, $\frac{d\sigma}{d\phi} = \left\{p^2 + \left(\frac{dp}{d\phi}\right)^2\right\}^{\frac{1}{2}}$;

and s and σ will be connected by an algebraical equation whenever $p^2 + \left(\frac{dp}{d\phi}\right)^2$ is a perfect square, as $\sigma - c = A_0\phi^{2n+1} + A_2\phi^{2n-1} + \&c. + A_{2n}\phi$, where A_0, A_2, &c. are definite functions of a_0, a_2, &c.

6441. (By the late Prof. CLIFFORD, F.R.S.)—It is known that, if four lines be given, the circles circumscribing the four triangles so formed meet in a point; and that, if five lines be given, the five points so belonging to their five tetragrams lie on a circle [MIQUEL's Theorem, see *Diary*

for 1861, p. 55]. Show that this series of propositions is interminable; so that, if $2n$ lines be given, they determine $2n$ circles that meet in a point; and if $2n+1$ lines be given, they determine $2n+1$ points that lie on a circle.

Solution by G. F. WALKER, M.A.; Prof. MATZ, M.A.; *and others.*

1. We know (SALMON's *Higher Curves*) that $(2n-1)$ tangents of a curve of the n^{th} class, among whose tangents the line at infinity counts for $(n-1)$, being given, that the locus of the focus is a circle; hence the $2n$ circles, so determined from $2n$ lines by leaving out one at a time, must meet in a point which is the focus of the curve of the same kind touching all the $2n$ lines.

2. Again, $2n+1$ tangents of a curve of the $(n+1)^{\text{th}}$ class, among whose tangents the line at infinity counts for n, being given, the locus of the focus is a circle. Hence, taking as a complex the curve of the n^{th} class touching $2n$ of the $(2n+1)$ lines and the point at infinity on the remaining one, the focus of the complex will be the focus of the curve, and the $(2n+1)$ points so determined lie on a circle.

6439. (By Prof. CROFTON, F.R.S.)—Prove that

$$\int e^{\frac{1}{2}x^2}\,dx = e^{\frac{1}{2}x^2}(x^{-1}+1\,.\,x^{-3}+1\,.\,3x^{-5}+1\,.\,3\,.\,5x^{-7}+\ldots).$$

Solution by J. HAMMOND, M.A.; G. HEPPEL, M.A.; *and others.*

If $u_n = \int e^{\frac{1}{2}x^2}x^n\,dx$, integrating by parts, we have

$$u_n = e^{\frac{1}{2}x^2}x^{n-1} - (n-1)\int e^{\frac{1}{2}x^2}x^{n-2}\,dx$$
$$= e^{\frac{1}{2}x^2}\{x^{n-1} - (n-1)\,x^{n-3} + (n-1)\,(n-3)\,x^{n-5} - \&c.\}.$$

The series in the question is obtained by putting $n = 0$.

6488. (By W. B. GROVE, B.A.)—If my son is to enter either the law or the church, he must go either to Oxford or to Cambridge. If he goes to Oxford without entering the law, or to Cambridge without entering the church, he will get a legacy at his uncle's death. He will fail to get the legacy under the following circumstances:—If he will not go to Oxford, and at the same time will not enter the church; or if he will not go to Cambridge, and at the same time will not enter the law—and under *no other* circumstances. I decide that he must enter either the law or the church: will he or will he not get the legacy?

Solution by ELIZABETH BLACKWOOD.

Let α, β, m, n, x respectively denote the statements: He will enter the *law*, He will enter the *church*, He will go to *Oxford*, He will go to *Cambridge*, He will get the *legacy*. The data will then be

$$(\alpha+\beta : m+n)\ (m\alpha'+n\beta' : x)\ (x'=m'\beta'+n'\alpha')\ (\alpha+\beta)\ (\alpha : \beta')\ (m : n'),$$

the last two factors being understood, though not expressed in the enunciation of the problem. We get, by inspection,

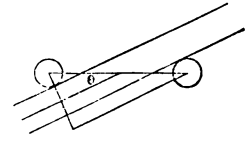
$$x' : (\alpha'\beta'+m+n)\ (m'+\alpha)\ (n'+\beta)\ (m'\beta'+n'\alpha')\ (\alpha+\beta)\ (\alpha'+\beta')\ (m'+n').$$

In the consequent of this implication, the last two factors are implied in the fourth, and may therefore be omitted. The product of the rest is zero. Hence $1 : x$, so that he will certainly get the legacy.

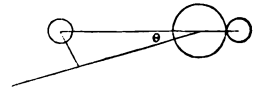
6447. (By CHRISTINE LADD.)—The width of a croquet hoop, the thickness of its wires, and the diameter of a ball are given: the ball being in a given position, show how to find the conditions that it may just be possible for it to go through the hoop, (1) straight, (2) by hitting one wire, (3) by hitting both wires; assuming that the angle of incidence is equal to the angle of reflection.

Solution by G. F. WALKER, M.A.; H. T. GERRANS, B.A.; *and others.*

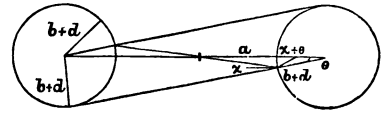
Let b be the radius of a wire, $2a$ the distance between the centres of the wires, $2d$ the diameter of the ball.



1. The line joining the centre of the ball to the centre of the distance $2a$ makes an angle $\sin^{-1}\frac{b+d}{a}$ with $2a$.



2. The tangent from the centre of the ball to a circle of radius $b+d$, with its centre at centre of nearer wire, $\sin^{-1}\frac{b+d}{2a-b-d}$ with the distance $2a$.



3. The tangent from the centre of the ball to the circle of radius $b+d$, with its centre at centre of nearer wire, makes an angle $\chi+\theta$ with the distance $2a$, χ and θ being given by the equations

$$\frac{b+d}{\sin(\chi-\theta)}=\frac{a}{\sin\chi},\qquad \frac{b+d}{\sin(\chi+\theta)}+\frac{(b+d)\sin\chi}{\sin(\chi+\theta)}=2a.$$

If the angle the tangent makes with $2a$ be less than this, the ball cannot get through without striking a wire more than once; and if the angle be less than $\sin^{-1}\frac{b+d}{2a}$, the ball cannot get through at all.

There is another limit, however; for, if $\chi+\theta$ be less than the result found from solving the equations

$$\frac{b+d}{\sin(\chi-\theta)}=\frac{2a}{\sin\chi},\quad \frac{b+d}{\sin(\chi+\theta)}+\frac{(b+d)\sin\chi}{\sin(\chi+\theta)}=2a,$$

the ball cannot get through, but will bound outwards from the nearer wire.

6479. (By Prof. WOLSTENHOLME, M.A.)—Prove that the asymptotic cone of a quadric $u=0$ is $A\left(\frac{\delta u}{\delta x}\right)^2+2F\frac{\delta u}{\delta y}\frac{\delta u}{\delta z}+\&c.=0$, and the equations of its axes are

$$\frac{A\frac{\delta u}{\delta x}+H\frac{\delta u}{\delta y}+G\frac{\delta u}{\delta z}}{\frac{\delta u}{\delta x}}=\frac{H\frac{\delta u}{\delta x}+B\frac{\delta u}{\delta y}+F\frac{\delta u}{\delta z}}{\frac{\delta u}{\delta y}}=\frac{G\frac{\delta u}{\delta x}+F\frac{\delta u}{\delta y}+C\frac{\delta u}{\delta z}}{\frac{\delta u}{\delta z}},$$

where $A=\frac{\delta^2 u}{\delta y^2}\frac{\delta^2 u}{\delta z^2}-\left(\frac{\delta^2 u}{\delta y\,\delta z}\right)^2$, $F=\frac{\delta^2 u}{\delta z\,\delta x}\frac{\delta^2 u}{\delta x\,\delta y}-\frac{\delta^2 u}{\delta y\,\delta z}\frac{\delta^2 u}{\delta x^2}$, &c.

Solution by Professor TOWNSEND, F.R.S.

Since, for every point (x, y, z) at which the asymptotic cone, or an axis of figure, meets a quadric, the tangent plane in the former case, and the normal line in the latter case, passes through the centre $x_0y_0z_0$ of the surface, we have consequently, between the two triads of coordinates, xyz and $x_0y_0z_0$, in the former case the relation

$$(x-x_0)\frac{du}{dx}+(y-y_0)\frac{du}{dy}+(z-z_0)\frac{du}{dz}=0,$$

and in the latter case the relations

$$(x-x_0):(y-y_0):(z-z_0)=\frac{du}{dx}:\frac{du}{dy}:\frac{du}{dz};$$

from which, since for every point xyz on the quadric, as appears at once from the known equations for the coordinates $x_0y_0z_0$ of its centre,

$$(x-x_0)D=A\frac{du}{dx}+H\frac{du}{dy}+G\frac{du}{dz},\quad (y-y_0)D=B\frac{du}{dy}+F\frac{du}{dz}+H\frac{du}{dx},$$
$$(z-z_0)D=C\frac{du}{dz}+G\frac{du}{dx}+F\frac{du}{dy},$$

where $D=abc+2fgh-af^2-bg^2-ch^2$; therefore &c., as regards both parts of the question.

Denoting by k the common value of the three equal ratios in the above equations for the axes of the surface, since then

$$(A-K)\frac{du}{dx}+H\frac{du}{dy}+G\frac{du}{dz}=0,\quad H\frac{du}{dx}+(B-K)\frac{du}{dy}+F\frac{du}{dz}=0,$$
$$G\frac{du}{dx}+F\frac{du}{dy}+(C-K)\frac{du}{dz}=0;$$

and since, consequently, $\begin{vmatrix} (A-K) & H & G \\ H & (B-K) & F \\ G & F & (C-K) \end{vmatrix} = 0$, we have, accordingly, for the determination of the three values of k, the well-known cubic determinant for the directions of the axes of the reciprocal surface. [See SALMON's *Geometry of Three Dimensions*, General Theory of Quadrics.]

If $u = 0$ represent originally a cone having its vertex at the origin, since then the two cones

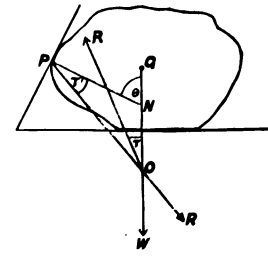
$$u = ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy = 0,$$
$$U = Ax^2 + By^2 + Cz^2 + 2Fyz + 2Gzx + 2Hxy = 0$$

are reciprocal polars to each other with respect to the point sphere having its centre at their common vertex $x^2 + y^2 + z^2 = 0$, and since the three surfaces have in consequence a common triad of principal planes, constituting in their case the Jacobian of the system, the equation of the triad for either of the two cones is accordingly $\begin{vmatrix} \frac{du}{dx} & \frac{du}{dy} & \frac{du}{dz} \\ \frac{dU}{dx} & \frac{dU}{dy} & \frac{dU}{dz} \\ x & y & z \end{vmatrix} = 0.$

[See SALMON's *Geometry of Three Dimensions*, General Theory of Quadric Surfaces, Note.]

6442. (By Prof. MINCHIN, M.A.)—A body of any shape, with a plane base, rests with this base on a rough horizontal plane; a heavy beam movable round a horizontal axis fixed in the plane rests at a single point against the body, the vertical plane through the beam containing the centre of gravity of the body. Show that limiting equilibrium of the system is impossible unless the normal to the surface of contact of the beam and body makes with the vertical an angle greater than the sum of the angles of friction between the body and the beam and the body and the ground.

Solution by D. EDWARDES; G. HEPPEL, M.A.; *and others.*



Let the total resistance of the ground meet the vertical through the centroid of the body in O. Let P be the point of contact of the beam with the body, PN the normal to the surface of contact, θ the inclination of PN to the vertical. Then the total resistance at P must pass through O, and make with the vertical an angle greater than λ; that is, we must have $\theta - \lambda' > \lambda$, or $\theta > \lambda' + \lambda$.

[Prof. MINCHIN states that he ought to have limited the question to equilibrium which is to be broken by the slipping of the body along the ground. It will be easily seen that, if the body can topple over, so that slipping takes place only at P, the result stated in the problem is not necessary.]

6417. (By C. J. Monro, M.A.)—The birds carry away the seed as fast as it falls, but the sower has the start of them by n grains. What is the probability that, after they have carried away x grains, y of those left are of the original n? (See *Origin of Species*, 5th ed., p. 381.)

I. *Solution by the* Rev. A. Law Watherston, M.A., F.R.A.S.

At any time, if p original grains be left, the chance of one being taken is $\frac{p}{n+1}$; $\frac{n+1-p}{n+1}$ being the chance of a fresh grain being taken.

Suppose, then, that a fresh grain is taken α times in succession; then an original one; then a fresh one β times in succession; then an original one; and so on. The chance in this instance is

$$\frac{1}{(n+1)^x} \cdot n \cdot \overline{n-1} \ldots (y+1) \cdot 1^\alpha \cdot 2^\beta \cdot 3^\gamma \ldots (n+1-y)^\lambda,$$

where $$\alpha+\beta+\gamma \ldots +\lambda = x-(n-y).$$

The total chance is therefore

$$\frac{1}{(n+1)^x} \cdot n \cdot \overline{n-1} \ldots (y+1)\, \Sigma \cdot 1^\alpha \cdot 2^\beta \cdot 3^\gamma \ldots (n+1-y)^\lambda,$$

the summation including all values of $\alpha, \beta, \gamma \ldots$ subject to the above condition.

The sum of the probabilities for all values of y being unity, we can find Σ by expanding $(n+1)^x$ in a series of factorials. Thus, putting $\phi(n)$ for $(n+1)^x$, we have

$$(n+1)^x = 1 + n \cdot \Delta\phi(0) + \frac{1}{2!} n^{(2)} \Delta^2\phi(0) + \ldots$$

The chance is therefore $\frac{1}{(n+1)^x} \cdot \frac{n!}{y!\,(n-y)!} \Delta^{n-y}\phi(0)$.

II. *Solution by* G. S. Carr, B.A.

For clearness suppose the n original grains to be white, and the rest black. Let $n-y=r$; then the probability of r black grains remaining is required.

A black grain falls; then the number of ways in which a grain can be carried off is $n+1$. Similarly, when a second grain falls. Thus, when x black grains have been sown, the number of ways in which x grains of either sort can have been carried away is $(n+1)^x$.

These are the unrestricted cases, the selection of any one grain at each juncture being the equally likely event with which the restricted events are now to be compared.

Let $B(r, x)$ stand for *the number of ways in which r black grains can remain after x have been sown.* Now, if, when r black grains remain, a black grain be added, then a black grain can be carried off in $r+1$ ways; and if, when $r-1$ black grains remain, a black grain be added, then a white grain can be carried off in $n-r+1$ ways; in each case *leaving r black grains.* Hence the following relation subsists

$$B(r, x) = (r+1)\, B(r, x-1) + (n-r+1)\, B(r-1, x-1) \ldots\ldots (1).$$

Applying this formula *ab initio*, observing that, from the nature of the

case, $B(r, x)$ is always zero when r is $> x$, and that $B(0, x)$ is always unity, we shall obtain the general value

$$B(r, x) = C_{nr}\left\{(r+1)^x - C_{r1}r^x + C_{r2}(r-1)^x - C_{r3}(r-2)^x + \ldots \pm 1\right\} \ldots (2),$$

the truth of which may be proved inductively by substituting from it the values of $B(r, x-1)$ and $B(r-1, x-1)$ in equation (1).

The probability of exactly r black grains remaining is

$$= B(r, x) \div (n+1)^x.$$

The probability of r black grains or fewer, that is, of y white grains or more, remaining is

$$= \left\{B(0, x) + B(1, x) + B(2, x) + \ldots + B(r, x)\right\} \div (n+1)^x,$$

with the values from formula (2) substituted in each case.

6461. (By H. McColl, B.A.)—Show that the weakest addition that must be made to the premises $A : a$, $B : b$, $C : c$, &c., to justify the inference $Q : q$, is the implication $Qq' : Aa' + Bb' + Cc' + \ldots$ [One of the problems solved by Mr. Venn, in the *Philosophical Magazine* for July, in illustration of his diagrammatic method, is a particular case of this general theorem.]

Solution by W. B. Grove, B.A.; Elizabeth Blackwood; *and others.*

To justify the inference $Q : q$, we must have $Qq' : 0$. Now, from the data $Aa' : 0$, $Bb' : 0$, &c., and the statement $Aa' + Bb' + Cc' + \ldots$ is weaker than any one or more of its terms, short of the whole; therefore the weakest additional datum necessary is

$$Qq' : Aa' + Bb' + Cc' + \ldots$$

6091. (By R. Tucker, M.A.)—If a point be taken on a certain line and normals be drawn from it to an ellipse, a circle can be drawn through the intersections of the normals with the ellipse. Find the envelop of this circle.

Solution by G. Eastwood, M.A.; *the* Proposer; *and others.*

The equation to the normal from (x', y') is

$$ax' \sin\phi - by' \cos\phi = c^2 \sin\phi \cos\phi;$$

that is, $$c^4z^4 - 2ac^2x'z^3 + (a^2x'^2 + b^2y'^2 - c^4)z^2 + 2ax'c^2z - a^2x'^2 = 0 \ldots\ldots\ldots(1),$$

where $$z \equiv \cos\phi.$$

Since the point (ϕ) is on the circle, we have

$$a^2\cos^2\phi + b^2\sin^2\phi + Aa\cos\phi + Bb\sin\phi + C = 0,$$

or $c^4x^4 + 2Aac^2x^3 + [A^2a^2 + B^2b^2 + 2c^2(C+b^2)]x^2$
$+ 2Aa(C+b^2)x + (C+b^2)^2 - B^2b^2 = 0$.....(2).

From identity of (1) and (2), we get

$A = -x'$, $ax'c^2 = Aa(C+b^2)$; therefore $C = -a^2$, $B^2b^2 = c^4 + a^2x'^2$.

Also, equating coefficients of x^2, we have $a^2x'^2 = b^2y'^2$; hence (x', y') must lie on one of the lines $by' = \pm ax'$.

The equation to the circle is $x^2 + y^2 - x'x + \frac{(c^4 + a^2x'^2)^{\frac{1}{2}}}{b} y - a^2 = 0$...(3).

For the envelop, we get

$$x'^2 = b^2c^4x^2 + [a^2(a^2y^2 - b^2x^2)], \quad c^4 + a^2x'^2 = a^2c^4y^2 + (a^2y^2 - b^2x^2);$$

and (3) becomes the quartic

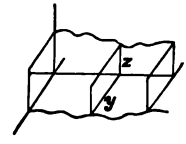
$$a^2b^2(x^2+y^2)^2 - 2a^4b^2(x^2+y^2) + a^6b^2 = c^4(a^2y^2 - b^2x^2).$$

6443. (By Prof. TAIT, M.A.)—Show that, whatever functions of x be represented by y and z, we have always $\frac{\int yz\,dx}{\int y\,dx} > e^{\frac{\int y \log z\,dx}{\int y\,dx}}$, all the integrals being taken between the same limits of x; and all the quantities involved being positive.

Solution by G. F. WALKER, M.A.; T. W. OPENSHAW, M.A.; *and others.*

Consider the volume bounded by the surfaces

$$y\,[=f(x)], \quad z\,[=\phi(x)].$$



The volume of this is $\int yz\,dx$, and the area of base is $\int y\,dx$. The given inequality is the same as

$$\log \frac{\int yz\,dx}{\int y\,dx} - \frac{\int y \log z\,dx}{\int y\,dx} > 0, \quad \log \bar{z} - \frac{\int y \log z\,dx}{\int y\,dx} > 0,$$

where $\bar{z}$ is the mean value of z for points in the area.

The second quantity is the mean value of $\log z$, which is of course less than the log of mean value of z. Hence the result.

6335. (By W. E. WRIGHT, B.A.)—If ds be an element of an arc of an ellipse whose semi-axes are a, b; and a', b' the semi-axes of the confocal hyperbola through the element; prove that $\frac{ds}{da'} = \left(1 + \frac{b^2}{b'^2}\right)^{\frac{1}{2}}$.

Solution by J. E. STEGGALL, M.A.; Prof. MATZ, M.A.; *and others.*

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1, \quad \frac{x^2}{a^2-\lambda^2}+\frac{y^2}{b^2-\lambda^2}=1,$$

$$x^2=\frac{a^2(a^2-\lambda^2)}{a^2-b^2}=\frac{a^2a'^2}{a^2-b^2}, \quad y^2=\frac{b^2b'^2}{a^2-b^2};$$

therefore $(dx)^2+(dy)^2=\frac{a^2(da')^2}{a^2-b^2}+\frac{b^2(db')^2}{a^2-b^2}$, but $a'da'=-b'db'$;

therefore $$ds^2=\frac{da'^2}{a^2-b^2}\left(a^2-b^2\frac{a'^2}{b'^2}\right)=da'^2\left(1+\frac{b^2}{b'^2}\right).$$

6435. (By W. E. WRIGHT, B.A.)—Eliminate α between the equations

$$(x^2\cos\beta+\cos\alpha)\cos\alpha+(x+\sin\alpha)\sin\alpha+x\sin\beta=0,$$
$$(x^2\cos\alpha+\cos\gamma)\cos\gamma+(x+\sin\gamma)\sin\gamma+x\sin\alpha=0.$$

Solution by C. MORGAN, B.A.; J. O'REGAN; *and others.*

Subtracting the equations, we get

$$x\cos\alpha\sin\tfrac{1}{2}(\beta+\gamma)=\cos\tfrac{1}{2}(\beta+\gamma).$$

Substituting in the first, we get

$$x\cos\beta\cot\tfrac{1}{2}(\beta+\gamma)+1+x\sin\alpha+x\sin\beta=0;$$

and therefore $$\left(x\frac{\cos\frac{1}{2}(\beta-\gamma)}{\sin\frac{1}{2}(\beta+\gamma)}+1\right)^2+\cot^2\tfrac{1}{2}(\beta+\gamma)=x^2,$$

$$x^2\cos\beta\cos\gamma+x(\sin\beta+\sin\gamma)+1=0.$$

6444. (By Prof. MATZ, M.A.)—Three points taken at random, one on each side of a plane triangle, are joined by straight lines; show (1) that the mean area of the triangle thus formed is one-fourth of the area of the triangle; and (2) generalize the problem for any number of points taken in different areas or volumes.

Solution by A. MARTIN, M.A.; J. O'REGAN; *and others.*

1. The mean will evidently be when each point is in its mean position, *i.e.*, at middle point of side, and therefore mean value is one-fourth of the area of the triangle.

2. Generally, if a number of points be confined to different areas or volumes which have no parts common, the mean value of the area or the solid formed by joining them, is the value of the area or solid formed by joining the centres of gravity of the areas or volumes to which the points are confined. [See *Reprint*, Vol. I., p. 50, Art. 6 of Solution of Quest. 1282.]

6066. (By Prof. WOLSTENHOLME, M.A.)—A heavy particle moves on the interior of a smooth paraboloid whose axis is vertical, and z_1, z_2 are its least and greatest vertical heights above the vertex, ρ the radius of absolute curvature of its path when at a vertical height z: prove that

$$\frac{4a(a+z)^3}{z}\,\frac{(z_1+z_2-z)^2}{\rho^2} = \frac{\{(z_1+z_2-z)[a(z_1+z_2-z)+z_1z_2]^2+az_1z_2(a+z)^2\}^2}{z(z_1+z_2-z)^2[a(z_1+z_2-z)+z_1z_2]^2+az_1^2z_2^2(a+z)^2}.$$

[As a test, suppose $z_1 = 0$, so that the path is plane, and we have $\frac{4(z_2-z)^2}{z}\,\frac{a(a+z)^3}{\rho^2} = \frac{[(z_2-z)^3a^2]^2}{z(z_2-z)^4a^2}$, or $\rho^2 = \frac{4(a+z)^3}{a}$, which is correct.]

Solution by the PROPOSER.

For the radii of absolute curvature at the highest and lowest points ρ_2, ρ_1, we have $\rho_1^2 = \frac{4az_2^2(a+z_1)}{z_2^2+az_1}$, $\rho_2^2 = \frac{4az_1^2(a+z_2)}{z_1^2+az_2}$;

and if $z_1 = z_2$, each of these becomes $4az_1$, the particle then describing a circle of radius $2(az_1)^{\frac{1}{2}}$.

The angle ϕ described about the axis by the projection of the particle on a horizontal plane is given by the equation

$$\frac{d\phi}{dz} = \frac{(z_1z_2)^{\frac{1}{2}}}{2.a^{\frac{1}{2}}}\,\frac{(a+z)^{\frac{1}{2}}}{z}\,\frac{1}{[(z-z_1)(z_2-z)]^{\frac{1}{2}}},$$

which is, as is natural, an elliptic function. The amount by which ϕ increases as the particle moves from its highest to its lowest point

$$= \left(\frac{z_1z_2}{a}\right)^{\frac{1}{2}}\int_0^{\frac{1}{2}\pi}\left(\frac{a+z_1\cos^2\theta+z_2\sin^2\theta}{z_1\cos^2\theta+z_2\sin^2\theta}\,d\theta\right)^{\frac{1}{2}},$$

and therefore lies between $\frac{1}{2}\pi\left(1+\frac{z_1}{a}\right)^{\frac{1}{2}}$ and $\frac{1}{2}\pi\left(1+\frac{z_2}{a}\right)^{\frac{1}{2}}$.

It will be found that this integral can be expressed in the form

$$\int_\alpha^\beta \frac{x\,dx}{2[x(x-1)(x-\alpha)(\beta-x)]^{\frac{1}{2}}}, \text{ where } \alpha = \frac{a+z_2}{z_2},\quad \beta = \frac{a+z_1}{z_1}.$$

6101. (By J. J. WALKER, M.A.)—If one side of a thin bi-convex lens is turned towards the sun, a well-defined image may be formed on a small screen facing that side, say at a distance of α^{-1}. The same being done with the other side, and the distance being now α'^{-1}, show that, if μ is the refractive index, the power of the lens is given by

$$2(2\mu-1)\phi = -(\mu-1)(\alpha+\alpha').$$

Solution by the PROPOSER.

The image referred to is formed by pencils which, undergoing refraction at the first surface of the lens, internal reflection at the opposite face, and

finally a second refraction on emergence from the first face, are brought to real foci in front of the lens. Let the reciprocals of the distances of the foci of the first refracted and reflected pencils from the lens be β, γ; the curvatures of the faces ρ, ρ' respectively, in the first position. Then

$$\mu\beta = (\mu-1)\rho, \quad \gamma+\beta = 2\rho', \quad \alpha-\mu\gamma = (1-\mu)\rho;$$

among which equations eliminating β, γ, $\alpha = -2(\mu-1)\rho + 2\mu\rho'$, an essentially positive expression, since ρ is negative, ρ' positive. When the lens is reversed, ρ and ρ' change signs, so that $\alpha' = 2(\mu-1)\rho' - 2\mu\rho$. Hence $\alpha+\alpha' = -2(2\mu-1)(\rho-\rho')$; but $\phi = (\mu-1)(\rho-\rho')$, and therefore $2(2\mu-1)\phi = -(\mu-1)(\alpha+\alpha')$.

The values of α, α' show that there will be two real images formed by the meniscus if the curvatures are in a ratio of greater inequality than $\mu : \mu-1$; one by the convexo-concave lens if of less inequality, the other being virtual. Those formed by the bi-concave lens are both virtual.

6454. (By W. H. H. Hudson, M.A.)—If NQ, CY be the perpendiculars from N, the intersection of the ordinate and directrix, and from C, the vertex, on the tangent at any point P of a catenary, and O be the point where the directrix meets the axis, prove that

$$QY : OC = \widehat{CP} - ON : PN.$$

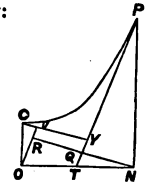
Solution by J. Hammond, M.A.; R. Leidhold, M.A.; *and others.*

The following are well-known properties of the catenary:

$$PN = OC\cosh\frac{ON}{OC}, \quad \widehat{CP} = OC\sinh\frac{ON}{OC},$$

$$\tan PTN = \sinh\frac{ON}{OC};$$

and the last may be written $\sec PTN = \cosh\frac{ON}{OC}$.



Now we have $QY = OC\sin PTN - ON\cos PTN$,

therefore $QY\cosh\frac{ON}{OC} = OC\sinh\frac{ON}{OC} - ON$, or $\frac{QY\,.\,PN}{OC} = \widehat{CP} - ON$.

[Otherwise: If ORV be parallel to PYQT, we have OV : OC = PQ : PN, and OR : NQ = ON : PN; but OV − OR = QY, NQ = OC, and PQ = $\widehat{CP}$; therefore QY : OC = $\widehat{CP}$ − ON : PN.]

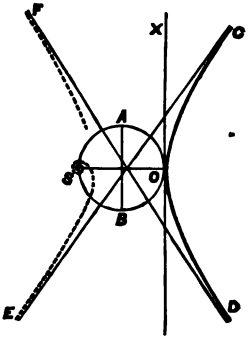
6460. (By J. W. Russell, M.A.)—Shew that, as the infinite branch of the curve $r = \frac{2a}{1-\theta^2}$ rolls on its initial tangent, the pole will describe a semicircle.

Solution by W. B. Grove, B.A.; C. Bickerdike, B.A.; *and others.*

As θ varies from -1 to 1, the portion DOC of the curve is described. Take OS, OX as axes. The locus of S, as DOC rolls on OX, is $y^2+(x-a)^2=a^2$,

that is, the circle SAOB.

By means of the equation $x=\frac{ar}{r-a}$, connecting the abscissa of S with the radius vector, we see that, when r is infinite, $x=a$; so that, as DOC rolls on OX, S describes the semicircle ASB.



6392. (By E. B. Elliott, M.A.)—A_1, A_2, A_3, A_4 are the vertices of a tetrahedron, P is the point whose volume coordinates with regard to it (*i.e.*, the ratios $PA_2A_3A_4 : A_1A_2A_3A_4$, &c.) are x_1, x_2, x_3, x_4, and O is any other point. Prove that, a_{12} being the length of the edge A_1A_2,

$$OP^2=x_1OA_1^2+x_2OA_2^2+x_3OA_3^2+x_4OA_4^2-\Sigma(a_{12}^2x_1x_2).$$

Solution by D. Edwardes; G. F. Walker, M.A.; *and others.*

Let (x', y', z', w') be the coordinates of O; then we have

$$-OP^2=\Sigma a_1a_2^2(x_1-x')(x_2-y') \quad \ldots\ldots(1),$$

and $$-OA_1^2=\Sigma a_1a_2^2x'y'-a_1a_2^2y'-a_1a_3^2z'-a_1a_4^2x'.$$

Hence the coefficient of x_1 in (1) is $-OA_1^2-\Sigma a_1a_2^2x'y'$,

so that $$-OP^2=-x_1OA_1^2-x_2OA_2^2-x_3OA_3^2-x_4OA_4^2+$$
$$\Sigma a_1a_2^2x_1x_2+\Sigma a_1a_2^2x'y'(1-x_1-x_2-x_3-x_4);$$

or, since $x_1+x_2+x_3+x_4=1$,

$$OP^2=x_1OA_1^2+x_2OA_2^2+x_3OA_3^2+x_4OA_4^2-\Sigma a_1a_2^2x_1x_2.$$

5526. (By Prof. Wolstenholme, M.A.)—Prove that, if p be positive and <1,

$$\int_0^1(x^p+x^{-p})\log(1+x)\frac{dx}{x}=\frac{\pi}{p\sin p\pi}-\frac{1}{p^2} \quad \ldots\ldots(1),$$

and $$\int_0^1(x^p+x^{-p})\log(1-x)\frac{dx}{x}=\frac{\pi}{p}\cot p\pi-\frac{1}{p^2} \quad \ldots\ldots(2),$$

of which (1) may be deduced from (2) by putting x^2 for x.

Solution by Prof. NASH, M.A.; J. HAMMOND, M.A.; *and others.*

$$\int_0^1 (x^p + x^{-p}) \log(1-x) \frac{dx}{x} = -\int_0^1 (x^p + x^{-p}) \left(1 + \frac{x}{2} + \frac{x^2}{3} + \ldots\right) dx$$

$$= -\left[\frac{x^{p+1}}{p+1} + \frac{x^{p+2}}{2(p+2)} + \frac{x^{p+3}}{3(p+3)} + \ldots - \frac{x^{-p+1}}{p-1} - \frac{x^{-p+2}}{2(p-2)} - \ldots\right]_0^1$$

$$= \frac{1}{p-1} - \frac{1}{p+1} + \frac{1}{2}\left(\frac{1}{p-2} - \frac{1}{p+2}\right) + \frac{1}{3}\left(\frac{1}{p-3} - \frac{1}{p+3}\right) + \&c.$$

$$= \frac{2}{p^2-1} + \frac{2}{p^2-2^2} + \frac{2}{p^2-3^2} + \&c.$$

$$= \frac{\pi}{p}\left\{\frac{2p\pi}{p^2\pi^2-\pi^2} + \frac{2p\pi}{p^2\pi^2-2^2\pi^2} + \frac{2p\pi}{p^2\pi^2-3^2\pi^2} + \&c.\right\}$$

$$= \frac{\pi}{p}\left\{\cot p\pi - \frac{1}{p\pi}\right\} = \frac{\pi}{p}\cot p\pi - \frac{1}{p^2}.$$

Putting x^2 for x in this integral, we have

$$\int_0^1 (x^{2p} + x^{-2p}) \log(1-x^2) \frac{2dx}{x} = 2\int_0^1 (x^{2p} + x^{-2p}) \log(1-x) \frac{dx}{x} + 2\int_0^1 (x^{2p} + x^{-2p}) \log(1+x) \frac{dx}{x};$$

$$\therefore \quad 2\int_0^1 (x^{2p} + x^{-2p}) \log(1+x) \frac{dx}{x} = \frac{\pi}{p}\cot p\pi - \frac{1}{p^2} - 2\left(\frac{\pi}{2p}\cot 2p\pi - \frac{1}{4p^2}\right)$$

$$= \frac{\pi}{p\sin 2p\pi} - \frac{1}{2p^2};$$

hence, writing p for $2p$, $\int_0^1 (x^p + x^{-p}) \log(1+x) \frac{dx}{x} = \frac{\pi}{p}\operatorname{cosec} \pi p - \frac{1}{p^2}$.

6401. (By R. KNOWLES, B.A., L.C.P.)—Prove that the locus of the middle points of normal chords to a parabola is the curve

$$(y^2 + 2a^2)^2 - 2axy^2 + 4a^4 = 0.$$

Solution by W. E. WRIGHT, B.A.; A. L. SELBY, B.A.; *and others.*

If (x', y') be one end of a normal chord, the other is

$$\frac{1}{4a}\left(\frac{8a^2}{y'} + y'\right)^2, \quad -\left(\frac{8a^2}{y'} + y'\right).$$

Let (x, y) be the middle of the chord, then we have

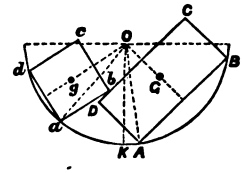
$$x = \frac{x'}{2} + \frac{1}{8a}\left(\frac{8a^2}{y'} + y'\right)^2, \quad y = \frac{y'}{2} - \frac{1}{2}\left(\frac{8a^2}{y'} + y'\right);$$

but $y'^2 = 4ax'$, therefore $x = x' + \frac{2a^2}{x'} + 2a$, and $y = -\frac{4a^2}{y'}$.

Using these values we obtain the stated equation.

4584. (By W. Siverley.)—Two uniform cylinders, of lengths $2a$ and $2b$, and the radius of each being r, rest in contact in a fixed smooth hemispherical bowl of radius R, their axes being in a vertical plane; find their position of equilibrium.

Solution by A. Martin, M.A.; *the* Proposer; *and others.*



Let ABCD be the larger cylinder, A and B touching the bowl at the lower and upper ends respectively, G its centre of gravity; *abcd*, *g* the like points in the other cylinder, *d* being the point of contact of the two cylinders, O the centre of the bowl; join OG, O*g*, cutting the cylinder in H and *h*, draw the radius OK vertical, and let $\angle GOK = \theta$, $\angle GOg = \beta$.

Taking moments about O, we have

$$2a\,[(R^2-a^2)^{\frac{1}{2}}-r]\sin\theta = 2b\,[(R^2-b^2)^{\frac{1}{2}}-r]\sin(\beta-\theta).$$

Whence $$\tan\theta = \frac{b\,[(R^2-b^2)^{\frac{1}{2}}-r]\sin\beta}{a\,[(R^2-a^2)^{\frac{1}{2}}-r] + b\,[(R^2-b^2)^{\frac{1}{2}}-r]\cos\beta}.$$

In the quadrilateral OH*dh*, we have

$$OH = (R^2-a^2)^{\frac{1}{2}}-2r, \quad Oh = (R^2-b^2)^{\frac{1}{2}}-2r, \quad hd = b,$$

and *h* and H are right angles; hence β may be determined.

6203. (By W. H. H. Hudson, M.A.)—If P be the population and W the wealth of an industrial community at a time t, (1) interpret $\frac{d}{dt}\left(\frac{W}{P}\right)$; and (2) show that, in such a community, if each person saves one-nth of his income, the average value of a person's work per unit of time is

$$n\frac{d}{dt}\left(\frac{W}{P}\right) + \frac{W}{P^2}\frac{dP}{dt}.$$

Solution by the Proposer.

$\frac{d}{dt}\left(\frac{W}{P}\right)$ is the average rate of increase (per unit of time) of a person's wealth, *i. e.*, the average person's rate of saving per unit of time; hence $\frac{d}{dt}\left(\frac{W}{P}\right)\delta t$ is the average person's saving in time δt, and $(n-1)\frac{d}{dt}\left(\frac{W}{P}\right)\delta t$ is the average person's expenditure in time δt. Now, let X be the average value of a person's work per unit of time, then $PX\delta t$ is the value of the whole community's work in time δt [or, strictly, the value is intermediate to $PX'\delta t$ and $(P+\delta P)X''\delta t$, where X′, X″ are the least and greatest values that X can have during the interval δt, the values differing from

$PX\delta t$ by quantities of the second order]; therefore $PX\delta t - \delta W$ is the expenditure of the whole community in time δt

$$= P \text{ multiplied by } (n-1)\frac{d}{dt}\left(\frac{W}{P}\right)\delta t;$$

$$\text{therefore } X = \frac{1}{P}\frac{dW}{dt} + n\frac{d}{dt}\left(\frac{W}{P}\right) - \frac{d}{dt}\left(\frac{W}{P}\right) = n\frac{d}{dt}\left(\frac{W}{P}\right) + \frac{W}{P^2}\frac{dP}{dt}.$$

Otherwise:—Each person in time δt must do work sufficient to provide himself with an *income* equal to n times his saving, and also must perform his share of providing average *wealth* for the increment of population;

$$\text{therefore} \qquad X\delta t = n\frac{d}{dt}\left(\frac{W}{P}\right)\delta t + \frac{1}{P}\cdot\frac{W}{P_4}\cdot\delta P.$$

6072. (By Prof. Cochez.)—Sur les côtés d'un triangle et dans le même sens on prend à partir des sommets de longueurs AD, BE, CF égales respectivement à $\frac{mAB}{n}$, $\frac{mBC}{n}$, $\frac{mAC}{n}$, on joint les points D, E, F. Déterminer (1) le rapport $\frac{DEF}{ABC}$; (2) le rapport $\frac{a^2+b^2+c^2}{a'^2+b'^2+c'^2}$, a, b, c étant les côtés de ABC, a', b', c' les côtés DEF; (3) Comment faut-il déterminer le rapport $\frac{m}{n}$ pour que le triangle DEF ait une surface minima?

Solution by G. Heppel, M.A.; E. Rutter; *and others.*

Suppose that a', b', c' represent the sides FD, DE, EF respectively. Then, by equating the values of cos C, as determined from the triangles EFC and ABC, we obtain

$$n^2c'^2 = (2m^2 - 3mn + n^2)a^2 + (2m^2 - mn)b^2 - (m^2 - mn)c^2;$$

hence, by symmetry and addition,

$$a'^2 + b'^2 + c'^2 = \frac{n^2 - 3mn + 3m^2}{n^2}(a^2 + b^2 + c^2).$$

Also $\frac{FEC}{ABC} = m(n-m)$; hence $\frac{DEF}{ABC} = \frac{n^2 - 3mn + 3m^2}{n^2}$,

the same ratio as before. This may be written

$\frac{1}{4} + 3\left(\frac{m}{n} - \frac{1}{2}\right)^2$; therefore DEF is a minimum when $\frac{m}{n} = \frac{1}{2}$.

6089. (By D. Edwardes.)—If A, B, C are the angles of a triangle, prove that

$$2\{\sin^2(A-B) + \sin^2(B-C) + \sin^2(C-A) + \sin(A-B)\sin(C-A)\cos A + \sin(B-C)\sin(A-B)\cos B + \sin(C-A)\sin(B-C)\cos C\}$$
$$= (\sin^2 A + \sin^2 B + \sin^2 C)(1 - 8\cos A\cos B\cos C).$$

Solution by G. HEPPEL, M.A.; W. B. GROVE, B.A.; *and others.*

$$2\sin^2(A-B)+2\sin^2(A+B)=4\sin^2A+4\sin^2B-8\sin^2A\sin^2B;$$

$$\therefore\quad U=2\sin^2(A-B)=4\sin^2A+4\sin^2B-2\sin^2C-8\sin^2A\sin^2B,$$

$$V=2\sin(C-A)\sin(B-C)\cos C=\sin(C-A)[\sin B+\sin(B-2C)]$$
$$=\sin^2C-\sin^2A-\tfrac{1}{4}\cos 4C+\tfrac{1}{4}\cos 2B$$
$$=\sin^2C-\sin^2A-\sin^2B+4\sin^2C-4\sin^4C;$$

therefore $U+V=3\sin^2A+3\sin^2B+3\sin^2C-8\sin^2A\sin^2B-4\sin^4C;$

$$\therefore \text{whole expression}=(\sin^2A+\sin^2B+\sin^2C)[9-4(\sin^2A+\sin^2B+\sin^2C)]$$
$$=(\sin^2A+\sin^2B+\sin^2C)(1-8\cos A\cos B\cos C).$$

6415. (By C. W. MERRIFIELD, F.R.S.)—A regular skeleton dodecahedron is made of 30 equal wires connected 3 by 3 with universal joints. It is then flattened symmetrically, so that two opposite pentagonal faces retain their shape—these two faces being connected with the corners of a regular decagon by 10 of the edges. Show that, in order to make the construction possible, each pentagon must be turned through $\frac{1}{2}\tan^{-1}2$.

I. *Solution by* C. A. SWIFT, B.A.; G. F. WALKER, M.A.; *and others.*

The angle turned through will be the vertical angle of a triangle whose base is one of the edges and sides are the radii of the circles circumscribing a pentagon and a decagon having the same sides.

If r_1, r_2 be these radii, a a side, and θ the required angle, we have

$$r_1=a.\frac{5^{\frac{1}{2}}+1}{2},\quad r_2=a.\frac{(5^{\frac{1}{2}}+1)}{(2.5^{\frac{1}{2}})^{\frac{1}{2}}};$$

$$a^2=a^2\left(\frac{5^{\frac{1}{2}}+1}{2}\right)^2+\frac{5^{\frac{1}{2}}+1}{2.5^{\frac{1}{2}}}a^2-2a^2.\frac{5^{\frac{1}{2}}+1}{2}.\frac{(5^{\frac{1}{2}}+1)^{\frac{1}{2}}}{(2.2^{\frac{1}{2}})^{\frac{1}{2}}}\cos\theta,$$

$$1=\frac{6+2.5^{\frac{1}{2}}}{4}+\frac{5+5^{\frac{1}{2}}}{10}-\frac{(5^{\frac{1}{2}}+1)^{\frac{1}{2}}}{(2.5^{\frac{1}{2}})^{\frac{1}{2}}}\cos\theta;$$

$$\frac{(5^{\frac{1}{2}}+1)^{\frac{1}{2}}}{(2.5^{\frac{1}{2}})^{\frac{1}{2}}}\cos\theta=\frac{60+20.5^{\frac{1}{2}}+20+4.5^{\frac{1}{2}}-40}{40}=\frac{6+2.5^{\frac{1}{2}}}{2.5^{\frac{1}{2}}};$$

therefore $\cos\theta=\dfrac{(5^{\frac{1}{2}}+1)^{\frac{1}{2}}}{(2.5^{\frac{1}{2}})},\quad \cos 2\theta=\dfrac{5^{\frac{1}{2}}+1}{5^{\frac{1}{2}}}-1=\dfrac{1}{5^{\frac{1}{2}}};$

and $\tan 2\theta=2,\quad \theta=\frac{1}{2}\tan^{-1}2.$

II. *Solution by the* PROPOSER.

Let A be the centre of figure of the two pentagons and the decagon. Now the figure must be twisted, so that the radii of the circles circumscribed to the pentagon and decagon may form a triangle of which the

third side is an edge of the solid; that is to say, if B is a corner of the pentagon, and C of the decagon, ABC will be a triangle of which

$$AB = \frac{1}{2\sin 36^\circ}, \quad BC = 1, \quad CA = \frac{1}{2\sin 18^\circ};$$

and A is the half-angle of revolution; for, if there were no twist, ABC would be in a straight line.

Now, $\cos A = \dfrac{AB^2+AC^2-BC^2}{2AB.AC} = \dfrac{\sin^2 18^\circ+\sin^2 36^\circ-4\sin^2 18^\circ\sin^2 36^\circ}{2\sin 18^\circ \sin 36^\circ}$;

substituting the values of these sines, we obtain

$$\cos A = \frac{\sqrt{5}-1}{2\sqrt{(5-2\sqrt{5})}}, \quad \cos 2A = \frac{1}{\sqrt{5}}, \quad \tan 2A = 2.$$

Hence, the angle turned through by one pentagon upon the other, supposing them to turn different ways, is arc tan 2. If they both turn one way, they would turn about the same angle; but, in either case, the angle A, which is the twist of each relatively to the decagon, would remain as above, namely $\frac{1}{2}$ arc tan A.

It does not appear that the skeleton dodecahedron can be flattened symmetrically about an axis joining two opposite corners.

6145. (By C. Leudesdorf, M.A.)—Find three square numbers such that, if each be subtracted from the sum of the other two, and the remainders doubled, the resulting numbers shall be perfect squares.

I. *Solution by* G. Heppel, M.A.; S. Tebay, B.A.; *and others.*

This is obviously the same thing as requiring the solution in whole numbers of the equations $x^2+y^2=w^2$, $y^2+z^2=u^2$, $z^2+x^2=v^2$, where u^2, v^2, w^2 are the numbers sought.

If $y=mx$ and $z=nx$, these equations are the same as

$$1+m^2=\square\ ; \quad 1+n^2=\square\ ; \quad m^2+n^2=\square \ldots\ldots\ldots (1, 2, 3).$$

From (1), (2), $m=\dfrac{1-h^2}{2h}$, $n=\dfrac{1-k^2}{2k}$; and from (3), $\dfrac{m}{n}=\dfrac{1-r^2}{2r}$;

therefore $$\frac{k(1-h^2)}{h(1-k^2)} = \frac{1-r^2}{2r};$$

and solving this as a quadratic to find r, we see that $h^2k^2(h^2+k^2)-4h^2k^2+h^2+k^2$ must be a square, a condition which is satisfied if $h^2+k^2=4$; and it is easy to show that in that case we have

$$h=\frac{4l}{l^2+1}, \qquad k=\frac{2(l^2-1)}{l^2+1}.$$

Assigning any value for l, we obtain h, k, m, n, x, y, z, u, v, w.

For example, if $l=3$; $h=\frac{6}{5}$; $k=\frac{8}{5}$; $m=-\frac{11}{60}$; $n=-\frac{32}{60}$; and, taking the least integral values of the rest, we have

$$x=240, \quad y=44, \quad z=117, \quad u=125, \quad v=267, \quad w=244.$$

II. *Solution by the* PROPOSER.

If x^2, y^2, z^2 are the numbers, $2(y^2+z^2-x^2) = a^2$, $2(z^2+x^2-y^2) = b^2$, $2(x^2+y^2-z^2) = c^2$; or, $a^2+4x^2 = b^2+4y^2 = c^2+4z^2 = a^2+b^2+c^2$.

These equations will be satisfied by

$$2x = 2z\cos\alpha + c\sin\alpha, \quad a = 2z\sin\alpha - c\cos\alpha, \quad 2y = 2z\cos\beta + c\sin\beta,$$
$$b = 2z\sin\beta - c\cos\beta, \quad 4z^2 = (2z\sin\alpha - c\cos\alpha)^2 + (2z\sin\beta - c\cos\beta)^2.$$

Let $\beta = \frac{1}{2}\pi - \alpha$; then the last equation gives $z = \frac{1}{4}c\operatorname{cosec} 2\alpha$, whence $x = \frac{1}{8}c(\frac{1}{2}\operatorname{cosec}\alpha + \sin\alpha)$ and $y = \frac{1}{8}c(\frac{1}{2}\sec\alpha + \cos\alpha)$; or if $\cot\frac{1}{2}\alpha = mn^{-1}$,

$$x = \tfrac{1}{8}c\left\{\frac{m^2+n^2}{8mn} + \frac{2mn}{m^2+n^2}\right\}, \quad y = \tfrac{1}{8}c\left\{\frac{m^2+n^2}{4(m^2-n^2)} + \frac{m^2-n^2}{m^2+n^2}\right\},$$
$$z = \frac{c}{16}\,\frac{(m^2+n^2)^2}{mn(m^2-n^2)}.$$

For integral values, put $c = 16mn(m^4-n^4)$; then

$$x = (m^2-n^2)(m^4+n^4+18m^2n^2), \quad y = 2mn(5m^4+5n^4-6m^2n^2),$$
$$z = (m^2+n^2)^3,$$

$m = 2$, $n = 1$ give $z = 125$, $y = 244$, $x = 267$.

6292 & 6325. (By J. MCDOWELL, M.A., and R. KNOWLES, B.A., L.C.P.)—If p_r denote the coefficient of x^r in the expansion of $(1+x)^n$, where n is a positive integer, prove that

$$p_1 - \tfrac{1}{2}p_2 + \tfrac{1}{3}p_3 - \ldots + \frac{1}{n}(-\tfrac{1}{2})^{n-1}p_n = 1 + \tfrac{1}{2} + \tfrac{1}{3} + \ldots + \frac{1}{n} \quad \ldots\ldots(1),$$
$$p_1 - 2p_2 + \ldots + (n-1)(-)^{n-2}p_{n-1} = 0 \quad \ldots\ldots(2),$$
$$\tfrac{1}{2}p_1 - \tfrac{1}{3}p_2 + \ldots + \frac{1}{n+1}(-)^{n-1}p_n = \frac{n}{n+1} \quad \ldots\ldots(3).$$

Solution by the Rev. D. THOMAS, M.A.; G. HEPPEL, M.A.; *and others.*

(1) Let s_n denote the series on the left-hand side of the equation; then

$$n(s_n - s_{n-1}) = 1-(1-1)^n = 1, \text{ therefore } s_n = s_{n-1} + \frac{1}{n};$$

therefore
$$s_n = 1 + \tfrac{1}{2} + \tfrac{1}{3} + \ldots + \frac{1}{n}.$$

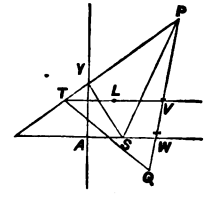
(2) If C_r^n denote the number of combinations of n things taken r at a time, $\Sigma_{r=1}^{r=n}(-)^{r-1}C_r^n = n\Sigma_{r=1}^{r=n}(-)^{r-1}C_{r-1}^{n-1} = n(1-1)^{n-1} = 0$.

(3) $\Sigma_{r=1}^{r=n}(-)^{r-1}\dfrac{C_r^n}{r+1} = \dfrac{1}{n+1}\Sigma_{r=1}^{r=n}(-)^{r-1}C_{r+1}^{n+1}$

$$= \frac{1}{n+1}\left\{\frac{(n+1)n}{2!} - \frac{(n+1)n(n-1)}{3!} + \ldots + (-)^{n+1}\right\} = \frac{n}{n+1}.$$

6429. (By R. Tucker, M.A.)—Given two tangents to a parabola, and the point on the axis through which the chord of contact passes; construct the parabola.

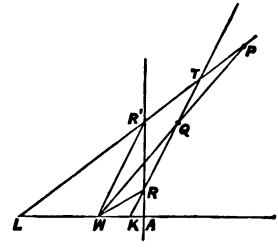
I. *Solution by* R. E. Riley, B.A.; R. Knowles, B.A., L.C.P.; *and others.*



Let TP, TQ be the given tangents; PQ the chord of contact; and W the point on the axis. Bisect PQ at V; then TV is a diameter. Draw SW parallel to TV; then SW is the axis. Make the angle TPS = PTV; then S is the focus; and if we draw SY perpendicular to TP, and through Y draw YA at right angles to axis, A will be the vertex, and the curve is determined.

The line PS is fixed in direction; and if L be the middle point of TV, L is the vertex of diameter TV; hence $4LS \cdot LV = PV^2$. Now, LV and PV being constant, LS is constant, and it seems that there would be only two positions of S, and therefore of A.

II. *Solution by the* Proposer.



The equation to PQ through P (m_1), Q (m_2) is $2m_1m_2x - y(m_1 + m_2) + 2a = 0$;

therefore $AK = -\dfrac{a}{m_1m_2}$.

The equation to WR (parallel to PT) is

$$y = m_1\left(x + \frac{a}{m_1m_2}\right),$$

therefore $AR = \dfrac{a}{m_2}$;

that is, R is the point on the tangent at the vertex where QT meets that line.

Hence, the construction is as follows:—through W draw straight lines parallel to the given tangents, meeting them in R, R'; RR' is the tangent at the vertex, whence the rest obviously follows.

6356. (By C. Leudesdorf, M.A.)—If 7 be divided by 5, giving 1 and 2 over; and then successively 21 by 5, giving 4 and 1 over; 14 by 5, giving 2 and 4 over; 42 by 5, giving 8 and 2 over, and so on until the figures recur, the result of placing all the quotients after the 7 and adding a decimal point suitably is $\cdot\dot{7}1428\dot{5}$, which is equal to $\frac{5}{7}$. Investigate the general form of the fractions $a : b$ which can be expressed as circulating decimals by dividing b by a in the manner explained with reference to the case of $\frac{5}{7}$.

I. *Solution by* W. H. WALENN; C. HARKEMA; *and others.*

Proposition I.—Decimal equivalents of all reciprocals of the form $(10a-1)^{-1}$ have the terminal figure of their period equal to 1. For, in all cases, the dividend that yields the first recurring figure is the same as that which commences the work; in the present instance 10. To effect this the remainder of the previous division must be 1. In $(10a-1)^{-1}$, the quotient figure which yields one for a remainder is always 1. Therefore 1 is the last figure of the period.

Proposition II.—The decimal equivalents of reciprocals of the form $(10a-1)^{-1}$ may be constructed by means of the multiplier a, this multiplier being used to obtain the next figure towards the left from the previous one as a multiplicand, and adding in the tens' digit that may be carried from the previous product.

For, by actual division,

$$(10a-1)^{-1} = (10a)^{-1} + (10a)^{-2} + \ldots + (10a)^{-n+1} + (10a)^{-n},$$

n being the number of figures in the recurring period of the decimal, and by Proposition I., $(10a)^{-n} = 1$, consequently $(10a)^{-n+1} = 10a$, and the series (commencing from the right-hand term) becomes

$$\ldots + (10a)^3 + (10a)^2 + 10a + 1.$$

By exactly similar reasoning this proposition or theorem may be extended to reciprocals having more than two figures in the denominator.

Proposition III.—Reciprocals of the form $(10a-3)^{-1}$ have 7 for the last figure of their period.

Since $7^2 = 49$, the multiplier which will reduce these reciprocals to a multiple of $(10a-1)^{-1}$ is 7. Therefore

(since $7 \cdot 1 = 7$), 7 is the last figure of the period.

In a similar way (since $3^2 = 9$), $(10a-7)^{-1}$ has 3 for the last figure of its period, and $(10a-9)^{-1}$ has 9 for the last figure of its period.

The modified multiplication treated of in Proposition II. will be found to be the inverse operation to the modified division in the question, by comparing the multiplicands, products, and carrying figures of the one with the quotients, dividends, and remainders of the other process.

The value of a in Proposition II. is also the value of the divisor referred to in the question.

In the case of $\frac{5}{7}$, the last figure is 5, since $\frac{5}{7} = 5 \cdot \frac{1}{7}$, and $7 \cdot 5 = 35$, the last figure of which is 5.

II. *Solution by the* PROPOSER.

Let the quotients be $q_1, q_2, \ldots$, the remainders $r_1, r_2, \ldots$, and the resulting decimal

$$D = \frac{b}{10} + \frac{q_1}{10^2} + \frac{q_2}{10^3} + \ldots;$$

then $b = aq_1 + r_1$, $10r_1 + q_1 = aq_2 + r_2$, $10r_2 + q_2 = aq_3 + r_3$, &c.;

therefore $$\frac{b}{10} + \frac{10r_1 + q_1}{10^2} + \ldots = \frac{aq_1 + r_1}{10} + \frac{aq_2 + r_2}{10^2} + \ldots;$$

therefore $$\frac{b}{10} + \frac{q_1}{10^2} + \ldots = \frac{aq_1}{10} + \frac{aq_2}{10^2} + \ldots,$$

$$D = a(10D - b), \quad D = \frac{ab}{10a-1}.$$

Now, if $D = \frac{a}{b}$, $a = \frac{b^2+1}{10}$, and b must be of one of the forms $10n+3$, $10n+7$, which gives $a = 10n^2+6n+1$ or $10n^2+14n+5$. Thus the general forms of the fractions $\frac{a}{b}$, which possess the property enunciated, are

$$\frac{10n^2+6n+1}{10n+3} \text{ and } \frac{10n^2+14n+5}{10n+7},$$

where n is any integer.

6218. (By Professor SYLVESTER, F.R.S.)—If

$E = a\delta_b + 2b\delta_c + 3c\delta_d + 4d\delta_e \ldots$, $F = a\delta_c + 3b\delta_d + 6c\delta_e + 10d\delta_f \ldots$,

$G = a\delta_d + 4b\delta_e + 10c\delta_f + 20d\delta_g \ldots$, $H = a\delta_e + 5b\delta_f + 15c\delta_g + 35e\delta_h \ldots$,

express $(E)^\mu$ as an algebraical function of E, F, G, H, &c.

Solution by W. J. CURRAN SHARP, M.A.

Let $E = E' + E''$, where E′ applies only to the function operated upon, and E″ only to the a, b, &c. involved in the operative symbols; then

$E'E \equiv F$, $(E'')^2E \equiv E''F \equiv G$, $(E'')^3E \equiv (E'')^2F \equiv E''G \equiv H + \&c.$;

therefore
$$(E)^2 = (E' + E'')\,E = (E')^2 + F,$$
$$(E)^3 = (E' + E')\,[(E')^2 + F] = (E')^3 + 3E'F + G,$$
$$(E)^4 = (E')^4 + 6\,(E')^2F + 4E'G + H, \ \ldots,$$
$$(E)^\mu = (E')^\mu + \frac{\mu(\mu-1)}{1.2}(E')^{\mu-2}F + \frac{\mu(\mu-1)(\mu-2)}{1.2.3}(E')^{\mu-3}G + \&c.$$

From these equations the values of $(E')^2$, $(E')^3$, &c., in terms of E, F, &c., may be successively determined, the first results being

$$(E')^2 = (E)^2 - F, \quad (E')^3 = (E)^3 - 3EF + 2G,$$
$$(E')^4 = (E)^4 - 6\,(E)^2F + 6F^2 - 4EG + 3H, \ \&c.$$

6008. (By Professor SYLVESTER, F.R.S.)—Prove that the equation

$$x^2y + x^2z + y^2x + y^2z + z^2x + z^2y = 0$$

is insoluble by positive or negative integer values of x, y, z.

Solution by W. J. CURRAN SHARP, M.A.

If
$$\frac{y}{z} + \frac{z}{y} = p, \quad \frac{z}{x} + \frac{x}{z} = q, \quad \frac{x}{y} + \frac{y}{x} = r,$$

p, q, r are each numerically > 2; and $p + q + r = 0$ by the given equation; also, by elimination, $pqr + 4 = p^2 + q^2 + r^2$; therefore $pqr + 4 > 0$, and con-

sequently pqr is positive; and hence two of the quantities p, q, r must be negative, and only one of x, y, z is negative. Let this be z; then, from the given equation, we have

$$\left(\frac{xy}{z^2}+1\right)(x+y) = -\frac{1}{z}(x^2+y^2) < -\frac{2xy}{z}, \qquad \left(\frac{xy}{z^2}+1\right)(x+y)+\frac{2xy}{z} < 0,$$

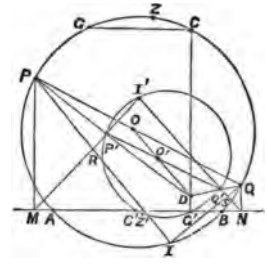
or

$$\left(\frac{x^2}{z^2}+\frac{2x}{z}+1\right)y+\frac{xy^2}{z^2}+x < 0,$$

which is impossible on the suppositions. Hence the equation is insoluble, &c.

[Prof. SYLVESTER states that this method of proof is quite distinct from his own, which proceeds upon the usual plan of showing that if the equation, or rather, a certain allied equation, is true for a given set of numbers, it will be true for another set with an inferior dominant (the dominant of a set of numbers meaning the greatest one of them without respect to sign). He believes Mr. SHARP's proof to be the first instance in questions of this kind where a direct method of negation has been employed, and if so, it may be regarded as constituting a new departure in the subject.]

6290. (See p. 38). II. *Solution by* Prof. TOWNSEND F.R.S.



Let A, B, C be the three vertices of the triangle; A′, B′, C′ the three middle points of its opposite sides; D its orthocentre; E, F, G the three second intersections with its circumscribing circle of the three parallels through A, B, C to its opposite sides; E′, F′, G′ the feet of three perpendiculars from A, B, C to its opposite sides; X, Y, Z the three points of trisection nearest to A, B, C of the three arcs AE, BF, CG of its circumscribing circle; X′, Y′, Z′ the three nearest to A′, B′, C′ of the three arcs A′E′, B′F′, C′G′ of its nine-point circle; O and O′ the centres of the circumscribing and nine-point circles respectively; P and Q any pair of diametrically opposite points on the circumscribing circle, P′ and Q′ the corresponding pair bisecting PD and QD, and of course also diametrically opposite, on the nine-point circle; PM and QN the two perpendiculars from P and Q upon the side AB of the triangle; PR and QS the two from the same upon the two lines of connection MI′ and NQ′ of M and N with P′ and Q′ respectively; I the intersection of PR and QS, and I′ that of P′M and Q′N;—then, since (*Reprint*, Vol. IV., p. 17) MP′ and NQ′ are the two tangents at the vertices of the two parabolas of the system of which P and Q are the foci, therefore PR and QS are the two axes of the same two parabolas; and since (see same) the two former MP′ and NQ′ intersect at right angles at the point I′ on the nine-point circle, and are tangents to the tricuspidal hypocycloid circumscribed to that circle at the three points X′, Y′, Z′ of its circumference, therefore the two latter PR and QS, being consequently the two parallels to NQ′ and MP′ at the doubles of their two distances in the two opposite directions from the external centre of similitude D of the two

circles, intersect at right angles at the point I on the circumscribing circle, and are tangents to the tricuspidal hypocycloid circumscribed to that circle at the three points X, Y, Z of its circumference; and therefore &c., as regards the envelop in question.

6158. (By Professor CROFTON, F.R.S.)—Find the mean distance of two points in a rectangle.

Solution by C. J. MONRO, M.A.

$$\frac{1}{a^2b^2}\int_0^a d\xi \int_0^a d\xi' \int_0^b d\eta \int_0^b d\eta' \,.\, [(\xi'-\xi)^2+(\eta'-\eta)^2]^{\frac{1}{2}n}, \;=\; \mathrm{M}_n \text{ say},$$

will give the mean n^{th} power of the distance. It is enough to say that, if we distinguish between opposite directions, the expression for a finite *number* of distances will be replaced, on passing to infinity, by

$$\frac{a}{d\xi}\cdot\frac{a}{d\xi'}\cdot\frac{b}{d\eta}\cdot\frac{b}{d\eta'},$$

and the expression for the sum of their n^{th} powers by the rest of the compound symbol.

The variables are present only by their differences $\xi'-\xi$ and $\eta'-\eta$, say x and y. Therefore we may eliminate ξ' and η', and, independently of the finite factors, integrate immediately as to ξ and η. For, while ξ is constant, $d\xi' = dx$; the limits of x are $\pm a$, but the finite factors being an even function, we may take the limits 0 and a and double the integral; the lower limit of ξ is the higher value of the two 0 and $-x$, namely 0, and its higher limit, the lower value of the two a and $a-x$, namely $a-x$. The like is true of η', η, and y. Integrating, then, as to ξ and η, and doubling twice for x and y, we have

$$\mathrm{M}_n = \frac{4}{a^2b^2}\int_0^a dx \int_0^b dy \,.\, (a-x)\,(b-y)\,(x^2+y^2)^{\frac{1}{2}n}.$$

When n is even and not negative, the integral is algebraical and rational. In general, if $x^2+y^2 = r^2$, and

$$\mathrm{R} = abr^{n+1} - (b\cos\theta + a\sin\theta)\,r^{n+2} + \cos\theta\sin\theta\,r^{n+3},$$

and $\alpha+\beta$ being $\frac{1}{2}\pi$, $a\sec\beta = b\sec\alpha$, $= (a^2+b^2)^{\frac{1}{2}}$, the result may be written

$$\mathrm{M}_n = \frac{4}{a^2b^2}\int_0^\beta d\theta \int_0^{a\sec\theta} \mathrm{R}\,dr + \frac{4}{a^2b^2}\int_\alpha^{\frac{1}{2}\pi} d\theta \int_0^{b\sec\theta} \mathrm{R}\,dr \ldots\ldots\ldots\ldots(1).$$

The former term is $\dfrac{4}{(n+3)\,a^2b^2}\displaystyle\int_0^\beta \left(\frac{a^{n+3}b}{n+2}\sec\theta^{n+2} - \frac{a^{n+4}}{n+4}\sec\theta^{n+3}\sin\theta\right) d\theta.$

If n is even or less than -1, this is a purely trigonometrical function; but if n is odd and not less than -1, it involves on reduction

$$\int_0^\beta \sec\theta\, d\theta, = \log \frac{b + (a^2+b^2)^{\frac{1}{2}}}{a}, \text{ say } G\beta.$$

When, as proposed, $n = 1$, we have

$$\frac{1}{a^2b^2}\left[\frac{a^4b}{3\,.\,2}(\sec^2\beta\sin\beta + G\beta) - \frac{a^5}{5\,.\,3}(\sec^2\beta - 1)\right],$$

which, if we put c^2 for a^2+b^2, is $\frac{c}{6} + \frac{a^2}{6b}G\beta - \frac{a^2c^3 - a^5}{15a^2b^2}$. The latter term of (1) is got by interchanging b and β with a and α; and, adding the two, we get

$$M_1 = \frac{c}{3} + \frac{1}{6}\left(\frac{a^2}{b}G\beta + \frac{b^2}{a}G\alpha\right) - \frac{c^5 - a^5 - b^5}{15a^2b^2}.$$

When a and α, or b and $\beta = 0$, the latter two terms destroy each other.

[See WILLIAMSON'S *Integral Calculus*, p. 348.]

6377. (By Professor GENESE, M.A.)—A quadrilateral linkage is capable in any position of having a circle inscribed in it. If one side be fixed, find the locus of the centre of the inscribed circle.

Solution by G. HEPPEL, M.A.; W. B. GROVE, B.A.; *and others.*

Let AB be the fixed side, $AD = b$, $BC = c$; and let the angles at A and B be 2θ and 2ϕ. Then, obviously, $CD = d = b + c - a$. Taking A as origin, the point D is $x = b\cos 2\theta$, $y = b\sin 2\theta$, and the point C is $x = a - c\cos 2\phi$, $y = c\sin 2\phi$; hence we have

$$d^2 = (b\cos 2\theta + c\cos 2\phi - a)^2 + (b\sin 2\theta - c\sin 2\phi)^2 = (b + c - a)^2,$$

or

$$ab\sin^2\theta + ac\sin^2\phi = bc\sin^2(\theta + \phi).$$

If the centre of inscribed circle is the point (x, y),

$$\sin\theta = \frac{y}{(x^2+y^2)^{\frac{1}{2}}},\quad \sin\phi = \frac{y}{[(a-x)^2+y]^{\frac{1}{2}}},\quad \sin(\theta+\phi) = \frac{ay}{(x^2+y^2)^{\frac{1}{2}}[(a-x)^2+y^2]}.$$

Substituting and reducing, we obtain, for the locus of the centre, the circle

$$\left(x - \frac{ab}{b+c}\right)^2 + y^2 = \frac{abcd}{(b+c)^2}.$$

5452. (By Professor SYLVESTER, F.R.S.)—Show that the fact of an equation $fx = 0$ of the n^{th} degree having r pairs of roots such that the product of each pair is a given constant, is conditioned by equating to zero r functions each of the $(n-2r+1)^{\text{th}}$ order in the constants; and, as a corollary to the case of $r = 1$, show that the resultant of $f(x, y)$ and $f(y, x)$ is a perfect square multiplied by the product of two linear factors.

Solution by W. J. Curran Sharp, M.A.

If $f(x) \equiv x^n + p_1 x^{n-1} + p_2 x^{n-2} + \ldots + p_n = 0$ be an equation which has r pairs of roots, such that the product of each pair $= c$. Then, by putting $x = c^{\frac{1}{2}} y$, it follows that the equation

$$c^{\frac{1}{2}n} y^n + p_1 c^{\frac{1}{2}(n-1)} y^{n-1} + \ldots + p_n = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots (1)$$

has r pairs of roots, such that the product of each pair is unity.

Let $$y^{2r} + a_1 y^{2r-1} + a_2 y^{2r-2} + \ldots + a_2 y^2 + a_1 y + 1 = 0 \quad \ldots\ldots\ldots\ldots\ldots (2)$$

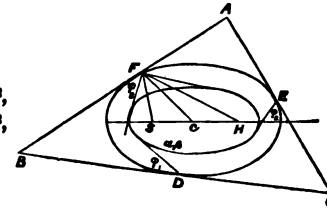
be the equation which gives these roots. The expression in (1) is divisible by that in (2), and therefore the remainder is 0 identically. This gives $2r$ equations involving the r quantities a_1, a_2, &c. to $n-2r+1^{\text{th}}$ power, and p_1, p_2, &c. to the first; and hence the resultants will be r functions of the $n-2r+1^{\text{th}}$ order in p_1, p_2, &c.

If $f(x, y) = 0$ and $f(y, x) = 0$ have a common root, the equation $f(x, 1) = 0$ must have a pair of roots whose product is unity, and the resultant must contain the function obtained above as a factor; it must also contain it again since $f(1, x) = 0$ has a similar pair, and this, with the product of p_0 and p_n, will produce the resultant.

6374. (By Professor Townsend, F.R.S.)—A conic being supposed to touch the three sides of a triangle, show that the three pairs of tangents to any confocal conic from the three points of contact are three pairs of common tangents to three circles of corresponding radii having their centres at the vertices of the triangle.

I. *Solution by* G. F. Walker, M.A.; W. B. Grove, B.A.; *and others.*

Let ab, $a\beta$ be the semi-axes of the given and confocal conics; $d_1 d_2 d_3$ the diameters parallel to BC, CA, AB; then



$$SF^2 + HF^2 + 2SF \cdot HF \cos 2\phi_3 = 4a^2,$$
$$2(CF^2 + CS^2) + 2d_3^2(1 - 2\sin^2\phi_3) = 4a^2,$$
$$a^2 + b^2 - (a^2 + \beta^2) = 2d_3^2 \sin^2\phi$$
$$= 2d_1^2 \sin^2\phi_1 = 2d_2^2 \sin^2\phi_2;$$
$$\therefore \quad d_3 \sin\phi_3 = d_1 \sin\phi_1 = d_2 \sin\phi_2;$$
$$\frac{d_3}{d_1} = \frac{BF}{BD}; \quad \text{therefore} \quad BF \sin\phi_3 = BD \sin\phi_1;$$

and similarly for others. $BF \sin\phi_3$ is perpendicular on tangent from B; therefore, &c.

II. *Solution by the* Proposer.

Denoting by α, β, γ the three angles made with the three sides a, b, c of the triangle by the three pairs of corresponding tangents to any conic S of

the confocal system, and by α', β', γ' the three for any other conic S' of the system; then, since, by a known property of confocal conics, $\sin\alpha : \sin\beta : \sin\gamma = \sin\alpha' : \sin\beta' : \sin\gamma'$, it follows at once that, if the property be true for any one conic S of the system, it is true for every other conic S' of the system; but it is evidently true for the point-pair consisting of the two foci E and F of the system; and therefore &c.

6422. (By E. B. Elliott, M.A.)—Prove that the envelop of the line

$$x\cos\phi + y\sin\phi = ae^{\phi} + be^{-\phi},$$

where ϕ is a variable parameter, is the evolute of the evolute of the evolute of its own evolute.

Solution by the Proposer; Prof. Matz, M.A.; *and others.*

$$\varpi \equiv x\cos\phi + y\sin\phi - ae^{\phi} - be^{-\phi} = 0$$

being the type tangent, $\frac{d\varpi}{d\phi} = 0$ is the type normal, that is, the type tangent of the evolute. So $\frac{d^2\varpi}{d\phi^2}, \frac{d^3\varpi}{d\phi^3}, \ldots$ equated to zero, give the type tangents of the second, third, and successive evolutes. Now

$$\frac{d^4\varpi}{d\phi^4} \equiv x\cos\phi + y\sin\phi - ae^{\phi} - be^{-\phi} \equiv \varpi \equiv \frac{d^8\varpi}{d\phi^8} \equiv \frac{d^{12}\varpi}{d\phi^{12}} \equiv \ldots.$$

Thus the type tangent of the 4^{th}, 8^{th}, ... $4n^{\text{th}}$ evolute is identical with that of the original curve. Wherefore, &c.

6186. (By T. R. Terry, M.A.)—If $a < 1$, and m and n are any positive integers, prove that

$$1 + \frac{(m-1)m}{1.n+1}\frac{a^2}{1-a^2} + \frac{(m-2)(m-1)m(m+1)}{1.2.(n+1)(n+2)}\frac{a^4}{(1-a^2)^2} + \ldots$$

$$= \frac{1}{(1-a^2)^{m-1}}$$

$$\times \left\{1 - \frac{n-m+1}{n+1}(m-1)a^2 + \frac{(n-m+1)(n-m+2)}{(n+1)(n+2)} \cdot \frac{(m-1)(m-2)}{1.2}a^4 - \ldots\right\}.$$

Solution by J. Hammond, M.A.; Rev. J. L. Kitchin, M.A.; *and others.*

Suppose, more generally, that m and n are any quantities whatever, and let $S = 1 + \frac{(m-1)m}{1.n+1}\frac{a^2}{1-a^2} + \frac{(m-2)(m-1)m(m+1)}{1.2.(n+1)(n+2)}\frac{a^4}{(1-a^2)^2} + \ldots;$

then $$S = \frac{\Gamma(n+1)}{\Gamma(n-m+1)\,\Gamma(m)}$$

$$\times \left\{ \frac{\Gamma(n-m+1)\,\Gamma(m)}{\Gamma(n+1)} + \frac{m-1}{1} \frac{\Gamma(n-m+1)\,\Gamma(m+1)}{\Gamma(n+2)} \frac{a^2}{1-a^2} \right.$$

$$\left. + \frac{(m-1)(m-2)}{1.2} \frac{\Gamma(n-m+1)\Gamma(m+2)}{\Gamma(n+3)} \frac{a^4}{(1-a^2)^2} + \&c. \right\}$$

$$= \frac{\Gamma(n+1)}{\Gamma(n-m+1)\,\Gamma(m)} \int_0^1 (1-x)^{n-m} x^{m-1} \left(1 + \frac{a^2}{1-a^2} x\right)^{m-1} dx;$$

or, changing x into $1-x$,

$$= \frac{\Gamma(n+1)}{\Gamma(n-m+1)\,\Gamma(m)} \int_0^1 (1-x)^{m-1} x^{n-m} \left(\frac{1-a^2x}{1-a^2}\right)^{m-1} dx$$

$$= \frac{\Gamma(n+1)}{\Gamma(n-m+1)\,\Gamma(m)} \frac{1}{(1-a^2)^{m-1}} \left\{ \frac{\Gamma(m)\,\Gamma(n-m+1)}{\Gamma(n+1)} - \frac{\Gamma(m)\Gamma(n-m+2)}{\Gamma(n+2)} \right.$$

$$\left. \times (m-1)a^2 + \frac{\Gamma(m)\,\Gamma(n-m+3)}{\Gamma(n+3)} \frac{(m-1)(m-2)}{1.2} a^4 - \&c. \right\}$$

$$= \frac{1}{(1-a^2)^{m-1}} \left\{ 1 - \frac{n-m+1}{n+1}(m-1)\,a^2 + \frac{(n-m+1)(n-m+2)}{(n+1)(n+2)} \right.$$

$$\left. \times \frac{(m-1)(m-2)}{1.2} a^4 - \&c. \right\}.$$

In the case where $n = 0$, $m = \frac{1}{2}$, we have from the definite integral, by an easy reduction, $$S = \frac{2(1-a^2)^{\frac{1}{2}}}{\pi} \int_0^{\frac{1}{2}\pi} \frac{d\theta}{(1-a^2\sin^2\theta)^{\frac{1}{2}}}.$$

The final value of S is equivalent to the ordinary expansion, of this integral; the original value of S gives another expansion, which is only convergent when $a^2 < 1-a^2$ or $a^2 < \frac{1}{2}$.

[Mr. TERRY remarks that this identity presented itself naturally in the consideration of the definite integral $\int_0^\pi \frac{\sin^{2n}x\,dx}{(1-2a\cos x + a^2)^m}$; but it admits however of the following simple proof:—

Let $$F\{\alpha, \beta, \gamma, x\} = 1 + \frac{\alpha.\beta}{\gamma.1}x + \frac{\alpha(\alpha+1).\beta(\beta+1)}{\gamma.(\gamma+1).1.2}x^2 + \ldots;$$

then it is well known that (CRELLE's *Journal*, Bd. XV., p. 52)

$$F\{\alpha, \beta, \gamma, x\} = (1-x)^{-\beta}\, F\left\{\beta, \gamma-\alpha, \gamma, \frac{x}{x-1}\right\},$$

whence, putting $\alpha = n-m+1$, $\beta = 1-m$, $\gamma = n+1$, $x = a^2$, we obtain the required result.]

6463. (By H. L. ORCHARD, M.A.)—Find the locus of the intersection of normals to the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ at the ends of a chord through the point $(\alpha, 0)$.

Solutions by R. KNOWLES, L.C.P.; R. E. RILEY, B.A.; *and others.*

The normals at the ends of the chord $l\frac{x}{a}+m\frac{y}{b}-1=0$ will meet in the point $x=-\frac{a^2-b^2}{a}\frac{l(m^2-1)}{l^2+m^2}$, $y=+\frac{a^2-b^2}{b}\frac{m(l^2-1)}{l^2+m^2}$.

The results of eliminating m, l between these two equations, are

$$\frac{l^2(l^2+1)^2}{(l^2-1)^2}\cdot\frac{b^2y^2}{(a^2-b^2)^2}=\left(l+\frac{ax}{a^2-b^2}\right)\left(1-\frac{alx}{a^2-b^2}\right),$$

$$\frac{m^2(m^2+1)^2}{(m^2-1)^2}\cdot\frac{a^2x^2}{(a^2-b^2)^2}=\left(m-\frac{by}{a^2-b^2}\right)\left(1+\frac{bmy}{a^2-b^2}\right).$$

If the chord pass through $(\alpha, 0)$, we have $l=\frac{a}{\alpha}$, and the locus is a conic.

Similarly, if it pass through $(0, \beta)$, we have $m=\frac{b}{\beta}$.

6433. (By J. W. MCKENZIE, B.A.)—Find, if possible, a value of n (other than $n=24$), so that the sum of the squares of the first n natural numbers may be a square.

Solution by G. HEPPEL, M.A.; G. EASTWOOD, M.A.; *and others.*

The sum of the squares of the first n natural numbers is $\frac{1}{6}n(n+1)(2n+1)$. Since n, $n+1$, $2n+1$ are all prime to one another; in order that the expression may be a square, each of the three must contain a square factor; and 6, or its factors 2 and 3, must be distributed among them as multipliers.

If $n=m^2$, then $n+1$ or m^2+1 cannot be of the form k^2 or $3k^2$ or $6k^2$, and the only possible case is

$$n=m^2,\ m+1=2k^2,\ 2m^2+1=3l^2 \quad \text{.....(A).}$$

If $n=2m^2$, $4m^2+1$ cannot be either l^2 or $3l^2$, and there is no possible case.

If $n=3m^2$, $3m^2+1$ cannot be either k^2 or $3k^2$, and there is no possible case.

If $n=6m^2$, then let $6m^2+1=k^2$, $12m^2+1=l^2$.....(B).

Taking case (B) first, the values of m, such that $6m^2+1$ is a square, are

0, 2, 20, 198, 1960, 19402, 192060, 1901198, &c.;

the law of formation being $u_n=10u_{n-1}-u_{n-2}$. The values of m, such that $12m^2+1$ is a square, are

0, 2, 28, 390, 5432, 75658, 1068780, &c.;

the law of formation being $u_n=14u_{n-1}-u_{n-2}$. The number 2 occurs in both these series, and in that case $n=6m^2=24$. But no other, less than a million, occurs in both, and therefore n cannot be less than six times the square of a million.

Taking case (A), we must first have $m^2+1=2k^2$; the values of m being

1, 7, 41, 239, 1393, 8119, 47321, 275807, and 1607521;

the law of formation being $u_n = 6u_{n-1} - u_{n-2}$; and, secondly, we must have the relation $2m^2 + 1 = 3l^2$.

$m = 1$ satisfies both conditions, and consequently gives a solution. The values of m that satisfy $2m^2 + 1 = 3l^2$, may, by solving the equations $2(m+l) = k(l+1)$, $k(m-l) = l-1$, be expressed generally thus

$$l = \frac{2+k^2}{2+4k-k^2}, \quad m = \frac{k^2+2k-2}{2+4k-k^2};$$

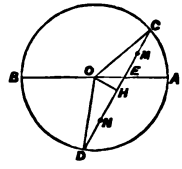
and, if $k = 4$, we have $m = 11$, $l = 9$. There is almost certainly no other solution in whole numbers. At any rate, if the numbers in the series last given be tested, they do not satisfy the relation $2m^2 + 1 = 3l^2$. Hence there is no solution under $n = (1{,}000{,}000)^2$.

[It is most probable that there is no solution other than $n = 24$.]

5708. (By Professor Seitz, M.A.)—Within a given circle a line is drawn at random in position and length; show that (1) the chance that the line intersects a given diameter is π^{-1}; and (2) the average length of the line is $\frac{3}{16}\pi r$.

Solution by the Proposer.

Let ABCD be the given circle, O its centre, AB the given diameter, and MN the random line. Produce MN, forming the chord CD, and draw OH perpendicular to CD. Let $OA = r$, $HM = x$, $MN = y$, $\angle COH = \theta$, and $\angle AEC = \phi$.



1. We have

$$OH = r\cos\theta, \quad CE = r(\sin\theta - \cos\theta\cot\phi),$$

and $$DE = r(\sin\theta + \cos\theta\cot\phi).$$

The number of ways the line MN can be drawn in the chord CD, so as to intersect AB, is $2CE \cdot DE = 2r^2(1-\cos^2\theta\operatorname{cosec}^2\phi)$, and the whole number of ways it can be drawn in the chord is $CD^2 = 4r^2\sin^2\theta$. The limits of θ are 0 and $\frac{1}{2}\pi$, and, in order that CD shall intersect AB, the limits of ϕ must be $\frac{1}{2}\pi - \theta$ and $\frac{1}{2}\pi + \theta$, and doubled. Hence the chance that MN intersects AB is

$$p = 2\int_0^{\frac{1}{2}\pi}\int_{\frac{1}{2}\pi-\theta}^{\frac{1}{2}\pi+\theta} 2r^2(1-\cos^2\theta\operatorname{cosec}^2\phi)\, r\sin\theta\, d\theta\, d\phi + \int_0^{\frac{1}{2}\pi}\int_0^{2\pi} 4r^2\sin^2\theta\, d\theta\, d\phi.$$

$$= \frac{3}{4\pi}\int_0^{\frac{1}{2}\pi}\int_{\frac{1}{2}\pi-\theta}^{\frac{1}{2}\pi+\theta}(1-\cos^2\theta\operatorname{cosec}^2\phi)\sin\theta\, d\theta\, d\phi$$

$$= \frac{3}{2\pi}\int_0^{\frac{1}{2}\pi}(\theta - \sin\theta\cos\theta)\sin\theta\, d\theta = \frac{1}{\pi}.$$

2. The limits of θ are 0 and $\frac{1}{2}\pi$; of ϕ, 0 and 2π; of x, $-r\sin\theta$ and $r\sin\theta$; and of y, 0 and $x + r\sin\theta = y_1$, and doubled; hence the average

length of MN is

$$\Delta = 2\int_0^{\frac{1}{2}\pi}\int_0^{\frac{1}{2}\pi}\int_{-r\sin\theta}^{r\sin\theta}\int_0^{\frac{1}{2}\pi} yr\sin\theta\,dx\,dy\,d\theta\,d\phi + \int_0^{\frac{1}{2}\pi}\int_0^{\frac{1}{2}\pi} 4r^3\sin^3\theta\,d\theta\,d\phi$$

$$= \frac{r}{2\pi}\int_0^{\frac{1}{2}\pi}\int_0^{2\pi}\sin^4\theta\,d\theta\,d\phi = r\int_0^{\frac{1}{2}\pi}\sin^4\theta\,d\theta = \tfrac{3}{16}\pi r.$$

6154. By Professor Sylvester, F.R.S.)—If the roots of a Quantic are all real, prove that the roots of its Hessian are all imaginary.

I. *Solution by the* Proposer.

Let $(a, b, c \ldots)(x, y)^n$ be a Quantic, all whose roots are real; then, by Newton's Rule, or by common sense, we know that b^2-ac must be positive: and obviously, if, when we write for y, $\lambda x+\mu y$, a, b, c become a', b', c' respectively, $b'^2-a'c'$ also must be positive for all real values of λ, μ.

But, on making this substitution, and then replacing λ, μ by y and $-x$, $b'^2-a'c'$ becomes the negative Hessian of the Quantic.

The Hessian, then, is always negative, and never changes its sign for any real value of $\frac{x}{y}$: consequently, its roots are all imaginary.

Cor. 1.—Since the roots of every derivative of the Quantic, obtained by differentiating in any number of times in respect to x, and any number of times in respect to y, must also be all real, it follows that the roots of the Hessians of all those derivatives must be all imaginary. The number of these, including the given Quantic, but stopping short at the derivatives of the second degree in x, y, because linear functions have no Hessian, will be $1+2+\ldots+(n-1)$ or $\frac{1}{2}(n-1)^2$. We thus obtain that number of essentially negative quantities, of which $(n-1)$ are the constants $c-b^2$, $(bd-c^2)$, ...; when these are left out, $\frac{1}{2}(n-2)(n-1)$ functions remain, of degrees ranging from $2n-4$ down to 2, all of which will have exclusively imaginary roots, in consequence of a single function of degree n having its roots exclusively real.

Cor. 2.—Since it is easy to show, for the Quantic $(a, b, c, d, e, f, g \ldots)(x, y)^n$ that the quantities $ae-4bd+3c^2$, $af-6bf+15ce-10d^2$ ... must be of the same sign as their last terms when the roots of the Quantic are all real, it follows, by the same reasoning as was applied to the Hessian, that if the roots of a Quantic are all real, the roots of each and every of its Quadri-covariants (*i.e.*, covariants of the second degree in the coefficients), and of the Quadri-covariants of its differential Derivatives, are all imaginary.

II. *Solution by* H. Stabbow, M.A.; W. J. C. Sharp, M.A.; *and others.*

Let $a_1, a_2, \ldots, a_n$ represent the n real roots of a quantic of the same degree in the variables. Its Hessian—expressed in terms of $a_1, a_2 \ldots$—will be (with sign changed)

$$\Sigma\,(a_1-a_2)^2\,(x-a_3)^2\,(x-a_4)^2\ldots(x-a_n)^2,$$

that is to say, it will consist of $\frac{1}{2}n\,(n-1)$ squares of the $2\,(n-2)^{\text{th}}$ degree in x.

These squares being all positive in our hypothesis, it is obvious that the Hessian cannot vanish for any real value of x, unless for that same value each of the squares reduce to 0 separately, which is impossible when, as supposed in the proposition, the quantities $a_1, a_2, \ldots$ are all unequal.

But the Hessian equated to 0 has $2\,(n-2)$ roots; these roots, then, must be all imaginary.

Should any number m of roots be equal, $a_1 = a_2 = \ldots = a_m$, then $\frac{1}{2}m\,(m-1)$ squares will evidently vanish and the remaining ones will contain as a common factor the binomial $(x-a_1)^{2\,(m-1)}$, so that in this case the number of imaginary roots of the Hessian will be reduced to $2\,(n-m-1)$.

[We get the same result, if we actually form the Hessian H of $(x-a)^m\,\phi\,(x, y)$, ϕ being a quantic of n dimensions in x, y.

For, denoting the first and second differential coefficients of ϕ by $\phi_1, \phi_2, \phi_{11}$, &c., we find

$$\mathrm{H} = \left\{\begin{array}{l}(\phi_{11}\phi_{22}-\phi_{12}^2)\,(x-ay)^2+2m\,(\phi_1\phi_{22}-a\phi_{11}\,.\,\phi_2+a\phi_1\phi_{12}-\phi_2\phi_{12})\,(x-ay)\\ \qquad +(m)\,(m-1)\,\phi\times(a^2\phi_{11}+2a\phi_{12}+\phi_{22})-m^2\,(a\phi_1+\phi_2)^2\end{array}\right\}$$

$$\times\,(x-ay)^{2(m-1)}$$

$$=\frac{n+m-1}{n\,(n-1)}\,.\,(x-ay)^{2(m-1)}$$

$$\times\left[(n+m)\,(\phi_{11}\phi_{22}-\phi_{12}^2)\,(x-ay)^2-\frac{m}{n-1}\left\{x\,(a\phi_{11}+\phi_{12})+y\,(a\phi_{12}+\phi_{22})\right\}^2\right],$$

in virtue of the known properties of homogeneous functions.

Since $\phi_{11}\phi_{22}-\phi_{12}^2$ is the Hessian of $\phi\,(x, y)$, and, as shown before, negative for any real value of x, the roots of the quantic within the brackets [ ] are necessarily all imaginary.]

6503. (By Sir James Cockle, F.R.S.)—Transform the equation (1) of Question 6404 into

$$\mathrm{Y}''+\left(\varpi\tan t-\frac{i\lambda}{\varpi}\right)\mathrm{Y}'+\tfrac{1}{4}\left\{1-\left(\varpi+\frac{\lambda}{\varpi}\right)^2-4\mathrm{N}\right\}\mathrm{Y}=0\ldots\ldots\ldots(\text{i.}),$$

where $i=(-1)^{\frac{1}{2}}$; show that (i.) admits of a continuous transformation whereby ϖ is changed into $\varpi-2$, $\varpi-4$, and so on; and notice the case of $\varpi=0$.

Solution by the Proposer.

1. Put $\mathrm{A}+\mathrm{E}=\mathrm{S}$, $\mathrm{A}-\mathrm{E}=\mathrm{D}$, $\tan x=\cos\theta+i\sin\theta$, and change the independent variable from x to θ; (1) becomes $y''+2qy'+ry=0$ .....(2), where $2q=(\mathrm{D}-1)\tan\theta+i\mathrm{S}$, $4r=(\mathrm{D}^2-\mathrm{Æ}^2)\,(\sec\theta)^2$; and if we write (i.) in the form

$$\mathrm{Y}''+2\mathrm{Q}\mathrm{Y}'+\mathrm{R}\mathrm{Y}=0\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(\text{ii.}),$$

then (see *Reprint*, IX., 105—112) $e^{\int q\,d\theta}\,\mathrm{Y}=e^{\int r\,d\theta}y$, provided that

$$\mathrm{R}-\mathrm{Q}^2-\mathrm{Q}'=r-q^2-q';$$

viz., if $1-4N-2\lambda+2i\lambda\tan\theta-(\varpi^2+2\varpi)(\sec\theta)^2$
$= S^2+(D-1)^2-2iS(D-1)\tan\theta+(1-\mathfrak{E}^2)(\sec\theta)^2$
identically. This gives $(\varpi+1)^2=\mathfrak{E}^2$, $\lambda=-S(D-1)$,
$1-4N=(S-D+1)^2=(2E+1)^2$, or $N=-E(E+1)$.
If $L=A(1-A)$, then $\lambda=L-N$.

2. Differentiating (ii.), and putting $Y'=U$, we get
$$U''+2QU'+\{R+\varpi(\sec\theta)^2\}U=0.$$
Assume $U''_2+2Q_2U'_2+R^2U_2=0$, then will $e^{\int Q_2 d\theta}U_2=e^{\int Q d\theta}U$, provided that $R_2-Q_2^2-Q'_2=R-Q^2-Q'+\varpi(\sec\theta)^2$. This is satisfied if Q_2, R_2 be derived from Q, R by changing ϖ into $\varpi-2$, or into $-\varpi$.

3. When $\varpi=0$, then $\mathfrak{E}^2=1$, and (2) is soluble if $D^2=1$, or again, if $S=0$. For, if $S=0$, then (2) transforms into $4u''+(D-1)^2u=0$. The independent variable t of the Question is herein replaced by θ.

5493. (By Professor SYLVESTER, F.R.S.)—If χ_q represent in general the number of linearly independent covariants of the degree q in the variables, and of a given order j in the constants belonging to a binary quantic of the degree i, prove that
$$\chi_0+2\chi_1+3\chi_2+4\chi_3+5\chi_4+\ldots\ldots=\frac{\Pi(i+j)}{\Pi i\,.\,\Pi j}.$$

[Here χ_0 represents the number of invariants, and the series stops spontaneously after the $(ij+1)^{th}$ term by all the subsequent χ's vanishing. Of course, when ij is even all the odd-indiced χ's, and when ij is odd, all the even-indiced χ's, vanish. The theorem as stated comprehends both of these cases in one.]

Solution by W. J. CURRAN SHARP, M.A.

The number of covariants of order j and weight of source $p=\frac{1}{2}(ij-q)$ is (SALMON'S *Higher Algebra*, p. 272) the difference of the coefficients of x^p and x^{p-1} in the expansion of
$$\frac{(1-x^{i+1})(1-x^{i+2})\ldots(1-x^{i+j})}{(1-x)(1-x^2)\ldots(1-x^j)}\equiv 1+A_1x+A_2x^2+A_3x^3+\&c.,$$
where A_1, A_2, &c. are all positive; and χ_p = the coefficient of $x^{\frac{1}{2}(ij-p)}$ in
$$\frac{(1-x^{i+1})(1-x^{i+2})\ldots(1-x^{i+j})}{(1-x^2)(1-x^3)\ldots(1-x^j)};$$
therefore $\chi_0+2\chi_1+3\chi_2+\&c.$ = the coefficient of $x^{\frac{1}{2}(ij)}$ in the product of this expression, and $1+2x^1+3x+\&c.$; $\left(i.e.\ \frac{1}{(1-x^1)}\right)$; *i.e.*, the coefficient of $x^{\frac{1}{2}(ij)}$ in
$$\frac{(1-x^{i+1})(1-x^{i+2})\ldots(1-x^{i+j})}{(1-x)(1-x^2)\ldots(1-x^j)}\times\frac{(1+x^1)^2}{1-x}=1+A_1+A_2+\ldots\ \&c.=\frac{\Pi(i+j)}{\Pi i\,.\,\Pi j},$$
since $A_r=A_{ij-r}$.

[A *Note* on this Question by the PROPOSER may be seen on p. 87 of Vol. XXXI. of the *Reprint*.]

5968. (By the late Prof. BENJAMIN PEIRCE, F.R.S.)—If two bodies revolve about a centre, acted upon by a force proportional to the distance from the centre, and independent of the mass of the attracted body, prove that each will appear to the other to move in a plane, whatever be the mutual attraction.

Solution by Professor ASAPH HALL, M.A.

At p. 91 of Vol. XXXIII. of the *Reprints* from the *Educational Times*, I find the above Question by Professor PEIRCE answered by Mr. MONRO. As I understand the Question, it seems to me correct. Suppose any number of particles attracted toward a centre according to the given law, and that the particles do not act on each other. Then the equations of motion for any particles are $\frac{d^2x}{dt^2}+kx=0$, $\frac{d^2y}{dt^2}+ky=0$, $\frac{d^2z}{dt^2}+kz=0$.

The solution of these equations shows that each particle moves in an ellipse which has its centre at the centre of this attracting force. For the relative motion of two particles, this equation will be

$$\frac{d^2(x-x_1)}{dt^2}+k(x-x_1)=0,\quad \frac{d^2(y-y_1)}{dt^2}+k(y-y_1)=0,$$
$$\frac{d^2(z-z_1)}{dt^2}+k(z-z_1)=0;$$

and since these equations have the same form as the first, we see that each particle will describe an ellipse around every other particle, the centre of the ellipse being at the particle. Now, let μ be the mutual attraction of two particles, r their distance apart, and let x, y, z be the coordinates of one particle with respect to another. The equations for the relative motion are

$$\frac{d^2x}{dt^2}+kx+\frac{\mu x}{r}=0,\quad \frac{d^2y}{dt^2}+ky+\frac{\mu y}{r}=0,\quad \frac{d^2z}{dt^2}+kz+\frac{\mu z}{r}=0.$$

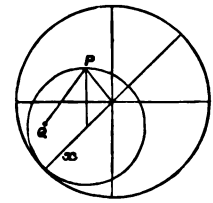
Multiply the second equation by z, and the third by y; we get, after integration, and two more similar combinations,

$$z.\frac{dy}{dt}-y.\frac{dz}{dt}=c,\quad x.\frac{dz}{dt}-z.\frac{dx}{dt}=c',\quad y.\frac{dx}{dt}-x.\frac{dy}{dt}=c''.$$

Hence we have $cx+c'y+c''z=0$; or the motion is in a plane, and this plane passes through the particle. [Mr. MONRO remarks that, according to Prof. HALL, $\frac{d^2x}{dt^2}:x=\frac{d^2x_1}{dt^2}:x_1$; that is, the *acceleration* towards the centre is independent of mass. Mr. MONRO had shown that the problem was correct when "force" was understood as acceleration.]

6248. (By the EDITOR.)—If two points P, Q be taken at random on the area of a vertical circle; show that the probability that the time of descent for a particle down the straight line PQ is less than that from P down the straight line of quickest descent to the circle, is $\frac{1}{2}$.

Solution by G. F. Walker, M.A.; Prof. Matz, M.A.; *and others.*



Let (r, θ) be the polar coordinates of P, and x the radius of the circle whose highest point is P, and touching the other O; then we have

$$(a-x)^2 = x^2 + r^2 - 2xr\sin\theta,$$

therefore $$2x = \frac{a^2-r^2}{a-r\sin\theta},$$

hence the required chance is

$$\iint \frac{\frac{x^2}{a^2}\,r\,d\theta\,dr}{\pi a^2} = \frac{1}{4\pi a^4}\iint\left[\frac{a^2-r^2}{a^2-r\sin\theta}\right]^2 r\,d\theta\,dr$$

$$= \frac{1}{4\pi a^4}\cdot 2\int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi}\int_0^a\left[\frac{a^2-r^2}{a-r\sin\theta}\right]^2 r\,d\theta\,dr = \frac{1}{2\pi}\int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi}\int_0^1\left[\frac{1-r^2}{1-r\sin\theta}\right]^2 r\,d\theta\,dr$$

$$= \frac{1}{2\pi}\int_0^{\frac{1}{2}\pi}\int_0^1 (1-r^2)^2\,r\,dr\,d\theta\left[\frac{1}{(1-r\sin\theta)^2}+\frac{1}{(1+r\sin\theta)^2}\right]$$

$$= \frac{1}{\pi}\int_0^{\frac{1}{2}\pi}\int_0^1 (1-r^2)^2 r\,dr\,d\theta\,\frac{1+r^2\sin^2\theta}{(1-r^2\sin^2\theta)^2} = \frac{1}{2\pi}\int_0^{\frac{1}{2}\pi}\int_0^1 (1-x)^2\,dx\,d\theta\,\frac{1+x\sin^2\theta}{(1-x\sin^2\theta)}$$

$$= \frac{1}{2\pi}\int_0^1\int_0^\infty (1-x)^2\,dx\,dy\,\frac{1+y^2+x}{(1+y^2-x)^2},\quad [y=\cot\theta]$$

$$= \frac{1}{2\pi}\int_0^1\int_0^\infty dx\,(1-x)^{\frac{1}{2}}\,dv\,\frac{1+x+(1-x)v^2}{(1+v^2)^2},\quad [y=(1-x)^{\frac{1}{2}}v]$$

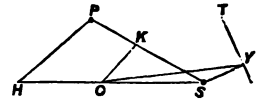
$$= \frac{1}{2\pi}\int_0^1\int_0^\infty dx\,(1-x^4)^{\frac{1}{2}}\,dv\,\frac{(1+v^2)+(1-v^2)\,x}{(1+v^2)^2}$$

$$= \frac{1}{2\pi}\int_0^1 dx\,(1-x)^{\frac{1}{2}}\int_0^\infty\left[\tan^{-1}v+\frac{vx}{1+v^2}\right] = \frac{1}{2\pi}\int_0^1 dx\,(1-x)^{\frac{1}{2}}\,(\tfrac{1}{2}\pi)$$

$$= \frac{1}{2\pi}\int^1 dx\,(x)^{\frac{1}{2}}\,(\tfrac{1}{2}\pi) = \frac{1}{2\pi}\cdot\frac{\pi}{2}\cdot\frac{2}{3} = \frac{1}{6}.$$

6480. (By Professor Genese, M.A.)—Given a tangent, a point, and a focus of a conic; the locus of the centre is a conic.

Solution by Charlotte A. Scott; C. Bickerdike, B.A.; *and others.*



Let P be the given point, S the focus, Y the foot of the perpendicular from S on the given tangent YT, H one position of the other focus, and C the corresponding centre. Then, (1) if S, H be on same side of YT, P lies on an ellipse with foci S, H; and (2) if S, H be on opposite sides of YT, P lies on an hyperbola with foci S, H; and, in each case, CY is the major semi-axis.

Let K be the middle point of PS; then $CK = \frac{1}{2}HP$; therefore (1) $CK + KS = \frac{1}{2}(HP + PS) = CY$, $\therefore CY - CK = KS$; thus the locus of C is an hyperbola with foci K, Y, and transverse axis = KS. Again, (2) $CK \sim KS = \frac{1}{2}(HP \sim SP) = CY$, $\therefore CK \mp CY = KS$ (the upper or lower signs being taken according as P and S are on the same or opposite sides of TY). Finally, the locus of C is a conic, with foci at K and Q, and major axis = $\frac{1}{2}SP$, and is a hyperbola or ellipse, according as S and P are on the same or opposite sides of TY.

6489. (By T. P. KIRKMAN, M.A., F.R.S.)—On a square chequer of n^2 small squares, stand $n^2 - 1$ counters marked 1, 2, 3 ... $(n^2 - 1)$, in any order S, covering all the squares except the last, at the right-hand corner; this last is occupied by a King, visible or invisible. The King, on whatever square he happens to stand, may exchange places only with any counter on a compartment collateral with his. The problem for solution is, to move the King about the board so that, when he has returned to his own final square, the order S shall be reduced to the natural order, the counters marked a, b, c... standing on the a^{th}, b^{th}, c^{th} ... squares. Required a rule whereby, on inspection of the order S, the solution can be proved possible or impossible.

[This problem is a generalized form of the famous American puzzle relating to 15 counters on 16 squares.]

Solution by the PROPOSER.

The given order S is a substitution made with n^2 elements, of which the final one, the King, is undisturbed. The only substitution which, operating on S, will restore the natural order, is S^{-1}, made with the same n^2 elements. If this restoration results from the movements of the King, their product as substitutions is equal to S^{-1}.

The King can make no excursion from home and back again, but by an even number of simple transpositions; for every step from home, whether vertical or horizontal, must be followed by an equal and parallel step in the opposite direction. It follows that S^{-1} is a positive substitution, *i.e.*, made by an even number of simple transpositions. Consequently S is positive, when the solution is possible.

If, then, the given S is negative, that is, if it has an odd number of circular factors, each of an even number of elements, a thing evident to inspection, the solution is impossible.

When S is positive, whatever be n^2, the King can dance round and head up every counter to its place, to rest there. When S is negative he can do this only for all the counters but the two last, which remain transposed.

An example on the chequer of nine squares is the following:—

Let S = 123586749, which is positive. King Nine has to take fourteen steps, which, written as substitutions, are

123956784	123456798	123496785	123956784
123456798	123956784	123956784	123496785
123496785	123459786	123456798	123459786
123456987		123456987	

Operating on the first by the second, on their product by the third, &c., the product of the fourteen is found to be $S^{-1} = 123846759$.

6499. (By J. L. McKenzie, B.A.)—If x_n and y_n be the n^{th} positive integral solution of $x^2 - Ny^2 = 1$, prove that $x_{n+p} = 2x_p x_n - x_{n-p}$, and $y_{n+p} = 2x\ y_n - y_{n-p}$.

Solution by Prof. Arnold Droz; L. W. Jones, B.A.; *and others.*

Soient x_1, y_1 les deux plus petites racines de l'équation $x^2 - Ny^2 = 1$, les racines x_n et y_n seront données par l'équation $(x_n + y_n N^{\frac{1}{2}}) = (x_1 + y_1 N^{\frac{1}{2}})^n$; donc $x_{n+p} + y_{n+p} N^{\frac{1}{2}} = (x_1 + y_1 N^{\frac{1}{2}})^{n+p} = (x_1 + y_1 N^{\frac{1}{2}})^n (x_1 + y_1 N^{\frac{1}{2}})^p$

$$= (x_n + y_n N^{\frac{1}{2}})(x_p + y_p N^{\frac{1}{2}}) = (x_n x_p + y_n y_p N) + (x_n y_p + x_p y_n) N^{\frac{1}{2}};$$

$$x_{n+p} = x_n x_p + y_n y_p N,\quad y_{n+p} = x_n y_p + x_p y_n \ldots\ldots\ldots\ldots\ldots\ldots (1).$$

On a de même

$$x_{n-p} + y_{n-p} N^{\frac{1}{2}} = (x_1 + y_1 N^{\frac{1}{2}})^{n-p} = \frac{(x_1 + y_1 N^{\frac{1}{2}})^n}{(x_1 + y_1 N^{\frac{1}{2}})^p} = \frac{(x_1 + y_1 N^{\frac{1}{2}})^n (x_1 - y_1 N^{\frac{1}{2}})^p}{(x_1 - N y_1^2)^p};$$

mais $x_1^2 - N y_1^2 = 1$, et $(x_1 - y_1 N^{\frac{1}{2}})^p = x_p - y_p N^{\frac{1}{2}}$, on aura

$$x_{n-p} + y_{n-p} N^{\frac{1}{2}} = (x_n + y_n N^{\frac{1}{2}})(x_p - y_p N^{\frac{1}{2}})$$

$$= (x_n x_p - y_n y_p N) + (x_p y_n - x_n y_p) N^{\frac{1}{2}};$$

$$x_{n-p} = x_n x_p - y_n y_p N,\quad y_{n-p} = x_p y_n - x_n y_p \ldots\ldots\ldots\ldots\ldots (2).$$

En comparant les équations (1) et (2), on obtient

$$x_{n+p} = 2x_n x_p - x_{n-p},\quad y_{n+p} = 2x_p y_n - y_{n-p}.$$

6219. (By Professor Townsend, F.R.S.)—A thin uniform spherical shell being supposed to hold in free equilibrium a material particle, situated at its centre of form, by the attraction of its mass for a single power, direct or inverse, of the distance; determine the groups of powers for which the equilibrium of the particle is respectively stable and unstable.

Solution by the Proposer.

Denoting by a the radius of the shell, by m its mass, by n the index of the law of force, and by R the internal attraction of the mass at the distance x from its centre; then, since, as may be easily shown,

$$R = (n+2)ma^n \left[\frac{n_0}{1.3}\left(\frac{x}{a}\right)^1 + \frac{n_2}{3.5}\left(\frac{x}{a}\right)^3 + \frac{n_4}{5.7}\left(\frac{x}{a}\right)^5 + \&c.\right],$$

where n_0, n_1, n_2, &c. are the several binomial coefficients for the index n, therefore, for a point so near the centre that all powers above the first of the ratio $x : a$ may be neglected, we have $R = \frac{1}{3}(n+2)ma^{n-1}x$; which being positive or negative with $n+2$, therefore, &c.

When $n+2 = 0$, that is, for the ordinary law of the inverse square of the distance, then, as is well known, $R = 0$ for all values of x internal to the shell.

6281. (By Prof. WOLSTENHOLME, M.A.)—Given two real foci and the real asymptote of a circular cubic ; prove that the locus of the other real foci is a circle whose centre lies on the given asymptote.

Solution by the Rev. F. D. THOMSON, M.A. ; W. GALLATLY, B.A. ; *and others.*

Let the given asymptote be the axis of y, and (a_1b_1) (a_2b_2) the two given foci, (XY) a third focus, then the equation of the cubic is

$$lr_1 + mr_2 = (l+m)r_3,$$

where r_1, r_2, r_3 are the distances from the three foci, and

$$r_1 = \{(x-a_1)^2+(y-b_1)^2\}^{\frac{1}{2}} = (x^2+y^2)^{\frac{1}{2}}\left(1 - \frac{2a_1x+2b_1y-a_1^2+b_1^2}{2(x^2+y^2)} + \dots\right),$$

and similarly for the others. Hence the equation of the real asymptote is

$$l(2a_1x+2b_1y-a_1^2-b_1^2)+m(2a_2x+2b_2y-a_2^2-b_2^2) = (l+m)(2Xx+2Yy-X^2-Y^2);$$

and since this coincides with the axis of y,

$$lb_1+mb_2 = (l+m)Y,\ l(a_1^2+b_1^2)+m(a_2^2+b_2^2) = (l+m)(X^2+Y^2);$$

and the locus is the *circle*

$$(X^2+Y^2-a_1^2-b_1^2)(b_2-Y) = (X^2+Y^2-a_2^2-b_2^2)(b_1-Y),$$

that is, the circle which passes through the two given foci, and has its centre on the given asymptote.

6483. (By R. TUCKER, M.A.)—From P, any point on the line $x = 2a$, tangents PT, PT′ are drawn to a parabola ; the normals at T, T′ (as is known) intersect on the curve (in Q). Prove that TT′ passes through a fixed point on the axis, on which axis also lies the centroid of Q, T, T′. the circle about PTQT′ passes through the vertex, and its envelope is a cubic curve ; and the orthocentre of QTT′ lies on the diameter through P and on a parabola.

Solution by D. EDWARDES ; J. O'REGAN ; *and others.*

Let m_1, m_2 be the parameters of the points T, T′. Let H be the orthocentre in question, O the "equidistant angle point," G the centroid. Since

P lies on the line $x = 2a$, we have $m_1 m_2 = 2$. The equation of TT', viz. $2x - (m_1 + m_2)y + 2am_1 m_2 = 0$, shows that TT' meets the axis in the fixed point $-2a$. Also, the coordinates of Q are easily found to be $\frac{1}{4}(a\lambda^2)$, $-a\lambda$, where $\lambda = 2(m_1 + m_2)$. Let x_P, y_P denote the coordinates of any point P. Then $3y_G = 2a(m_1 + m_2) - a\lambda = 0$; therefore G lies on the axis. Again, the equation of circle about PTQT' is

$$(x - x_Q)(x - x_P) + (y - y_Q)(y - y_P) = 0, \text{ or } ax\lambda^2 - 2ay\lambda + 4(2ax - x^2 - y^2) = 0;$$

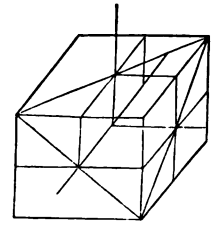
its envelope is therefore the cubic $ay^2 + 4x(x^2 + y^2 - 2ax) = 0$. Let A be the vertex, then the equation of AQ is $\lambda y + 4x = 0$, and of AP is $y = \frac{1}{2}(m_1 + m_2)x = \frac{1}{4}\lambda x$; therefore PAQ is a right angle, and the circle passes through A. Now $x_G = \frac{1}{6}a(\lambda^2 - 8)$, $y_G = 0$, $x_0 = a(1 + \frac{1}{8}\lambda^2)$, $y_0 = -\frac{1}{4}a\lambda$; therefore, by a known geometrical theorem,

$$x_H = \tfrac{1}{4}a\lambda^2 - 6a, \quad y_H = \tfrac{1}{4}a\lambda = y_P.$$

Hence H lies on the diameter through P, and on the parabola $y^2 = a(x + 6a)$.

6414. (By Elizabeth Blackwood.)—If x, y, z be each taken at random between 1 and -1, find the chances (1) that $x + y$, $y + z$, $z + x$ are all three positive, (2) that $x + y - xy$, $y + z - yz$, $z + x - zx$ are all three positive.

I. *Solution by* G. F. Walker, M.A.; H. Fortey, M.A.; *and others.*



1. The chance will be the ratio of the volume of the cube

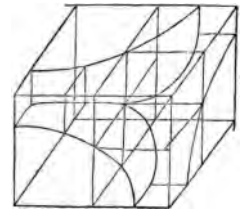
$$x = \pm 1, \ y = \pm 1, \ z = \pm 1,$$

to the positive side of the planes

$$y + z = 0, \ z + x = 0, \ x + y = 0,$$

to the whole volume; and therefore chance is

$$\frac{3 \cdot \frac{1}{2} + 1}{(2)^3} = \frac{1}{4}.$$



2. The chance will be the ratio of the volume cut off from the cube to the positive side of the hyperbolic cylinders

$$yz = y + z, \ zx = z + x, \ xy = x + y.$$

The finite intersections of these are along the planes $y = z$, $z = x$, $x = y$;
and the volume cut off is

$$1 + 3\int_{-1}^{0} \left(1 - \frac{x}{x-1}\right)^2 dx = 1 + 3\int_{-1}^{0} -\frac{1}{x-1}$$

$$= 1 + 3(1 - \tfrac{1}{2});$$

and chance is

$$\frac{1 + 3(1 - \frac{1}{2})}{(2)^3} = \frac{5}{16}.$$

II. *Solution by* W. B. Grove, B.A.; Colonel J. H. Fry; *and others.*

The chance (1) required is $\dfrac{AQ \iiint dx\,dy\,dz}{A \iiint dx\,dy\,dz}$,

where A denotes the statement $x_{1'.2}\ y_{1'.2}\ z_{1'.2}$ (Table I.),

and $Q = p(x+y)\,p(y+z)\,p(z+x) = y_3\ z_{3.4}$.

Table I.

$x_1 = 1$	$y_1 = 1$	$z_1 = 1$
$x_2 = -1$	$y_2 = -1$	$z_2 = -1$
	$y_3 = -x$	$z_3 = -y$
	$y_4 = x$	$z_4 = -x$

The limits of integration are readily found by Mr. McColl's method, and the result is

$$AQ = x_{1'.0}\ y_{4'.3}\ z_{1'.3} + x_{0'.2}\ y_{1'.4}\ z_{1'.4} + x_{1'.0}\ y_{1'.4}\ z_{1'.4},$$

so that chance (1) $= \frac{1}{3}(1+\frac{1}{3}+\frac{2}{3}) = \frac{1}{2}$.

Table II.

$x_1 = 1$	$y_1 = 1$	$z_1 = 1$
$x_2 = -1$	$y_2 = -1$	$z_2 = -1$
$x_3 = \frac{1}{2}$	$y_3 = -\dfrac{x}{1-x}$	$z_3 = -\dfrac{y}{1-y}$
	$y_4 = \frac{1}{2}$	
	$y_5 = x$	$z_4 = -\dfrac{x}{1-x}$

For chance (2), A will have the same as before; but (referring to Table II.)

$$Q = p(x+y-xy)\,p(y+z-yz) \times p(z+x-zx) = y_3 z_{3.4}.$$

Hence $AQ = x_{1'.2}\ y_{1'.2.3}\ z_{1'.2.3.4}$, that is,

$$AQ = x_{1'.3}\ y_{1'.4}\ z_{1'.2} + x_{1'.3} y_{4'.2}\ z_{1'.3} + x_{3'.0}\ y_{5'.3}\ z_{1'.3} + x\ \ y_{1'.3}\ z_{1'.4} + x_{3'.0}\ y_{1'.5}\ z_{1'.4};$$

so that chance (2) $= \frac{1}{3}\{\frac{1}{2} + \log 2 + (1 - \log 2) + \frac{1}{3} + \frac{1}{4}\} = \frac{5}{18}$.

6434. (By W. H. Besant, M.A.)—Having given two circles of radii R and r, and that the distance between their centres is $(R^2 - 2Rr)^{\frac{1}{2}}$, prove that an infinite number of triangles can be drawn which shall be inscribed in one circle and circumscribed about the other.

I. *Solution by* D. Edwardes; Prof. Nash, M.A.; *and others.*

Let the equations of the circles be

$$S \equiv x^2 + y^2 - r^2 = 0, \quad S' \equiv (x-\delta)^2 + y^2 - R^2 = 0.$$

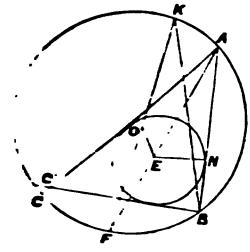
Calculating the invariants, we have

$$\Theta = \frac{\delta^2 - R^2}{r^4} - \frac{2}{r^2}, \quad \Delta = -\frac{1}{r^4}, \quad \Theta' = \frac{\delta^2}{r^2} - 1 - \frac{2R^2}{r^4}.$$

The condition $\Theta^2 = 4\Delta\Theta'$ leads to

$$(\delta^2 - 2r^2 - R^2) + 4(\delta^2 r^2 - r^4 - 2r^2R^2) = 0, \quad \text{or} \quad \delta^2 = R^2 \pm 2rR.$$

I. Solutions of the Problems.



[illegible] W. Lawrence, M.A.—Prove that the surface

$$[illegible] = a^3$$

[illegible] and find the equation of the generating [illegible].

[illegible] W. Pressland, M.A.; J. O. Kelly, B.A.; *and others.*

$$[illegible] = [illegible]$$

$$= [illegible].$$

[illegible] $X = 0$, we get

$$[illegible] = \frac{2a^3}{3\sqrt{3}};$$

[illegible] and the curve referred to its [illegible].

[illegible] R. J. Thomson, M.A.)—A conic cuts a cubic in 6 [illegible] at these points meet the [illegible].

[illegible] G. Perelli; *and others.*

[illegible] and let a', a'' be the 1st and 2nd [illegible] of a', a is the point where the conic of

5-pointic contact at a meets the cubic again (SALMON's *Higher Curves*, Art. 155).

But a', b', c', d', e', f' lie on one conic, for the conic $abcdef$ taken twice, and the conic through 5 points a', b', c', d' e' form a system of the 6th degree passing through $3 \times 6 - 1$ points, and therefore passing through one other fixed point on the cubic (SALMON, Art. 156). But the 6 tangents, aa', bb', &c. form another system of 6th degree through the same 17 points, and therefore f' is also on the conic. The same reasoning proves that a'', b'', c'', d'', e'', f'' lie on a conic, and also that the residuals of a, a'', b, b'', &c. lie on another conic.

[If a, b, c be in a straight line, the same reasoning shows that a'', b'', c'' are in a straight line, and therefore the residuals of a, a'', b, b'', &c. are in a straight line, which furnishes a solution to Quest. 6210.]

6342. (By Prof. GENESE, M.A.)—Prove that the envelop of the directrix of a parabola (1) inscribed in, or described about, a triangle, is a conic; (2) touching two straight lines and passing through a point, or passing through two points and touching a straight line, is a quartic.

Solution by the PROPOSER.

The results are given incorrectly; the following is the method employed. With the usual notation, the equation to the directrix of a parabola represented by the general equation, is $2Gx + 2Fy = A + B$. That the directrix should pass through a given point P is therefore a line-condition. (WHITWORTH's *Trilinear Coordinates*, Chap. xxv.) That the curve is a parabola is another line-condition. With these conditions, (α) *one* conic only can be drawn to touch three straight lines; (β) *two* conics to touch two straight lines and pass through a given point; and (γ) *four* conics to touch one straight line and pass through two points, or to pass through three given points. Therefore, as regards the envelop, from any point P there can be drawn one, two, or four directrices of the systems (α), (β), (γ) respectively. The envelopes are therefore (α) a point, (β) a conic, or (γ) a curve of the fourth class.

6468. (By A. MARTIN, M.A.)—An auger hole is made through the centre of a sphere; show that the chance that the volume removed exceeds one-n^{th} of the sphere is $1-\left[1-\left(1-\frac{1}{n}\right)^{\frac{3}{2}}\right]^{\frac{1}{3}}$.

Solution by PROF. MATZ, M.A.; Prof. Evans, M.A.; *and others.*

Put a = the radius of the sphere; b = the radius of the auger-hole; and V = the volume removed. Then, we readily find $V = \frac{4}{3}\pi\left[a^3-(a^2-b^2)^{\frac{3}{2}}\right]$; therefore $b = a\left[1-\left(1-\frac{1}{n}\right)^{\frac{3}{2}}\right]^{\frac{1}{3}}$, and chance $= \frac{a-b}{a}$ = &c.

6362. (By E. B. Elliott, M.A.)—If m lines, straight or curved, are drawn across an enclosed portion of area; prove that, if n_1 be the number of simple internal intersections of these lines with one another or themselves, n_2 the number of triple points, n_3 of quadruple, &c., formed by them, the number of separate areas into which they divide the inclosure is

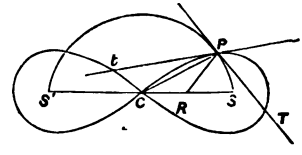
$$1 + m + n_1 + 2n_2 + 3n_3 + \ldots.$$

Solution by W. B. Grove, B.A.; E. Rutter; *and others.*

Consider first two lines which do not intersect, then divide any enclosed area into 3 parts. Now let them intersect; the number of parts becomes 4, therefore every point of simple intersection that is introduced increases the number of parts by 1. Similarly, it may be seen that every triple point introduced increases the number of parts by 2, and so on. Also m lines not intersecting divide the area into $(m+1)$ parts, hence the total number of parts is $1 + m + n_1 + 2n_2 + 3n_3 + \ldots.$

6457. By E. B. Elliott, M.A.)—P is one of the intersections of a lemniscate with the circle on the intercept SS′ between its foci as diameter. Prove that the two curves cut at an angle of 60°, and that the bisector of the angle SPS′ is also the bisector of the angle between the two tangents at P.

Solution by E. W. Symons, M.A.; D. Edwardes; *and others.*



Let the equation of the lemniscate be $r^2 = a^2 \cos 2\theta$. At the point where it meets the semicircle in SS′, we must put $r = \frac{1}{2}a\sqrt{2}$, whence $\theta = 30°$. Now, $\angle CPZ = 90° - 2\angle SCY$; hence the angle between the tangents is $2\angle SCY$, that is, 60°.

Again, PT being the tangent to the circle at P, and PR the bisector of the angle between the tangents,

$$\angle SPR = 60° - SPT; \text{ but } SPT = 90° - \angle SPC = 15°;$$

therefore $\angle SPR = 45°$, therefore PR bisects also the angle SPS′.

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